The Sturm-Liouville problem of a second order nonlinear ordinary differential equation

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Abstract

The work is devoted to the study of the Sturm-Liouville boundary value problem

$$\begin{gathered} y'' + f(t, y, y') = 0, \\ \alpha_{11} y(\alpha) + \alpha_{12} y'(\alpha) = 0, \quad \alpha_{21} y(\beta) + \alpha_{22} y'(\beta) = 0 \end{gathered}, \tag{1}$$

, which has been investigated by a number of authors. Conditions for the existence of a solution are provided, and estimates of the solution are given. The authors demonstrate that the main theorem implies, in particular, certain results from the works of A. L. Teptin and N. N. Yuberev, and L. S. Rakovshchik. Moreover, the existence of a solution to the problem is established for the case $\alpha_{11}\alpha_{12} > 0$; $\alpha_{21}\alpha_{22} < 0$, which has not been previously considered by other authors. Bibliography: 7 items.

Full Text

1. Introduction and Preliminary Results

In 1967, B. M. Abramov and A. Ya. Khokhryakov [?] investigated the boundary value problem for the second-order differential equation:

$$y'' + f(t, y, y') = 0$$

subject to the boundary conditions:

$$\begin{gathered} a_{11} y(a) + a_{12} y'(a) = 0 \\ a_{21} y(\beta) + a_{22} y'(\beta) = 0 \end{gathered}$$

In their work [?], they assumed that $a_{12} > 0$ and $a_{21}a_{22} < 0$. Under these conditions, they established existence and uniqueness theorems. Specifically, they analyzed the case where $a_{21}a_{22} < 0$ and $a_{12} > 0$. In this section, we extend these results to the n-th order case $y^{(n)} + f(t, y, y', \dots) = 0$ under various boundary conditions.

Consider the linear system:

$$y' + Ay = Q, \quad My(a) + Ny(\beta) = 0$$

where A is a constant matrix. Let $f(t)$ be a continuous function on $[a, T]$ for $a < T$. We define the operator L and the corresponding boundary value problem as:

$$x'(a) + Nx(\beta) = 0 \quad (1.4)$$

Following the methodology of Abramov and Khokhryakov [?], the solution on the interval $[a, \beta^*]$ can be represented via the Green' s function $K(t, s)$ as:

$$x(t) = \int_a^\beta K(t, s) f(s) ds \quad (1.5)$$

Lemma 1

If $\tau > 0$, $\alpha > 0$, $\tau \neq \alpha$, and $s < 0$, then the expression $\exp(-Bs)$ satisfies specific positivity conditions. Specifically, we assume the boundary matrix coefficients satisfy $b_{11} > 0$, $b_{12} < 0$, $b_{21} = 0$, and $b_{22} > 0$ (1.6).

Lemma 2

If $\tau > 0$ and $s > 0$, then $\exp(-Bs) > 0$ (1.7). The Green' s function $K(t, s)$ for the operator $Lx = f(t)$ can be constructed such that:

$$x(t) = \int_a^\beta K(t, s) f(s) ds + K(t, a)[M + NK(a, \beta)]^{-1} NK(\beta, s) \quad (1.8)$$

From (1.4) and (1.8), we derive the relation:

$$y[M + NK(a, \beta)] = \int_a^\beta G(t, s) f(s) ds \quad (1.9)$$

where the kernel $\Gamma(t, s)$ is defined for $a < t < s < \beta$ and $a < s < t < \beta$ using the fundamental solution $K(t, s) = \exp[-B(t - s)]$.

2. Green' s Function Properties and Estimates

For the case where $A - \Lambda = 0$ (1.14), the components of the Green' s matrix $\Gamma(t, s)$, denoted as $\gamma_{11}, \gamma_{12}, \gamma_{21}, \gamma_{22}$, are derived in equations (1.12)-(1.15). We observe that for $\tau > \alpha$, the following inequality holds:

$$\exp[-(\tau - \alpha)(\beta - t)] > 1 - r \quad (1.16)$$

This implies that $\gamma_{12}(t, s) > 0$ for $t > s$ on the interval $[a, \beta]$. Furthermore, we establish that:

$$\frac{\exp[-(\tau-\alpha)(\beta-t)]-r}{\tau-\alpha} > 0 \quad (1.17)$$

provided that $\tau - \alpha > 1$ and $0 < r < 1$. At the point $t = a$, the expression simplifies, and we can determine the interval length $\beta - a$ required to maintain the sign of the Green' s function.

Consider the problem:

$$x' - Ax = 0, \quad x(a) + Nx(\beta) = 0 \quad (2.1)$$

where $x > 0$, $\alpha < 0$, and the boundary coefficients satisfy $a_{11}a_{12} < 0$ and $a_{21}a_{22} > 0$. Using the fundamental solution (1.10), the components of the Green' s function for (2.1) satisfy:

$$\gamma_{11} > 0, \quad \gamma_{12} < 0, \quad \gamma_{21} = 0, \quad \gamma_{22} < 0 \quad (2.6)$$

for all $t, s \in [a, \beta]$.

3. Existence of Solutions via Monotone Iterations

We now consider the nonlinear problem:

$$Mx(a) + Nx(\beta) = 0, \quad A\dot{x} + Qx = f(t, x) \quad (4.3)$$

By applying the transformation $x = Ty$, the system can be rewritten in a form suitable for the application of fixed-point theorems in a space with a cone. Let L_2 be the operator associated with (4.3). We assume the existence of upper and lower solutions v_1 and v_2 such that:

$$\begin{aligned} \dot{v}_1 + Bv_1 \le T^{-1}f(t, Tv_1) \\ \dot{v}_2 + Bv_2 \ge T^{-1}f(t, Tv_2) \end{aligned} \quad (4.6),\ (4.7)$$

The solution can be represented by the integral operator:

$$x(t) = \int_a^\beta \Gamma(t, s)T^{-1}[QTx(s) - f(s, Tx(s))]ds \equiv Gx \quad (4.8)$$

Under the condition (4.5), the operator G is monotonic. If $v_1(t) \le x(t) \le v_2(t)$, then the sequence of iterations $w(t) = Gx(t)$ converges to the solution of (4.3). This confirms the existence of a solution $u(t)$ satisfying $v_1(t) \le u(t) \le v_2(t)$ for $t \in [a, \beta]$.

4. Second-Order Equations and Differential Inequalities

For the second-order case, we consider the equation:

$$y'' - b_1y' - a_1y = f(t, y, y') \quad (4.16)$$

with boundary conditions $y(a) = Ky(\beta) = 0$. We assume the existence of functions $u(t)$ and $v(t)$ that satisfy the differential inequalities:

$$\begin{aligned} u'' - b_1 u' - a_1 u &\leq -\phi(t) \\ v'' - b_1 v' - a_1 v &\geq \phi(t) \end{aligned} \quad (4.14),\ (4.15)$$

where $\phi(t) > 0$. If the function f satisfies the growth condition $|f(t, y, u)| \leq \phi(t)$ (4.12), then there exists a solution $y(t)$ such that $u(t) \leq y(t) \leq v(t)$ for $t \in [a, \beta]$. This result generalizes the work of Abramov and Khokhryakov to cases where $a_{12} > 0$ and $a_{21}a_{22} < 0$.

5. Uniqueness and Lipschitz Conditions

Finally, we address the uniqueness of the solution. Suppose $f(t, y, u)$ satisfies a Lipschitz condition:

$$|f(t, y_1, u_1) - f(t, y_2, u_2)| \leq l_1|y_1 - y_2| + l_2|u_1 - u_2| \quad (5.1)$$

Let $\gamma(t, s)$ and $\gamma'_t(t, s)$ be the Green' s function and its derivative for the linear problem (5.2). The solution to the nonlinear problem (4.1) can be expressed as:

$$y(t) = \int_a^\beta \gamma(t, s) f(s, y(s), y'(s)) ds \quad (5.5)$$

A sufficient condition for the existence and uniqueness of the solution is:

$$l_1 \max_{t\in[a,\beta]} \int_a^\beta |\gamma(t, s)| ds + l_2 \max_{t\in[a,\beta]} \int_a^\beta |\gamma'_t(t, s)| ds < 1 \quad (5.6)$$

By applying the contraction mapping principle to (5.7)-(5.9), we conclude that the boundary value problem has a unique solution in the specified domain.

References

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Figures

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DIFFERENTIAL EQUATIONS

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ESTIMATES OF THE SOLVABILITY REGION FOR A SYSTEM OF EQUATIONS OF A MEMBRANE NET CYLINDRICAL SHELL IN THE CASE OF THE FIRST BOUNDARY VALUE PROBLEM

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Shells formed by mutually intersecting layers of absolutely flexible elastic threads in absolutely flexible elastic threads, have numerous applications in mechanical engineering.

A net shell, pre-stressed by air pressure inside its inner cavity, serves as the power framework of a pneumatic tire. In works [1, 2], a method for solving the calculation of net shells is developed, consisting in splitting the full stressed state in a thin momentless shell into a stressed state bate caused by symmetrical loading (internal air pressure) — the initial state, state, and a stressed state caused by additional loads, while linear differ- ente: — the initial state, and a stressed state caused by additional loads, while linear differential equilibrium equations for small deformations from the initial state under the action of additional loads are obtained in the linear approximation.

Equations for determining displacements and forces in threads for small deformations of shells from the initial state, containing in the general case of the shell rotation variable coefficients depending on the shape of the shell in the initial state, are significantly simplified upon transition to a cylindrical shell. Such a transition can be completed if one of the curvatures in the shell rotation is assumed to be equal to zero, while the initial shape of the shell section is a circle, and the coefficients of the system of equilibeiium equations become constant.

The results obtained in the study of a cylindrical shell can be used as a first approximation to a shell rotation in the case where the curvature of the shell in case where the curvature of the shell in one of the directions is sufficiently small.

At the same time, since the main properties of the problems, such as the type of equations, boundary conditions, right-hand side, are proserved upon transition to a cylindrical shell, the study of this particular case is of decisive methodological significance.

Initially, the equilibrium equations for the net shell were obtained in work [1] using the assumption of the non-extensibility of the length of the threads in the deformation process (conditions of non-extensibility of threads). This essential asseumption leads to the fact that the system of deformation equations turns out to be of non-elliptic type, with the directions along which the threads are located being the characteristics of the system. When solving boundary value problems for systems of differential equations of non-ellipnic type, serious difficulties arise when they justifying the legitimacy of using known solution methods (methods of expansion in trigonometric series, various variational methods), since a priori even the existence of a solution is not guaranteed.

Figure 1: Figure 1

In the future, the work [2] carries out a generalization to the case of an arbitrary characteristic of the thread (dependence of force in it on deformation) and obtains equilibrium equations for a net shell with extensible threads.

Using the example of cylindrical shell equations, it is shown that the introduction of thread extensibility significantly facilitates the study of the coprectness of the boundary value problem formulation (the concept of correctness, as is well known, means the existence and uniqueness of at least a generaliznod solution depending contineously on the right side of the system).

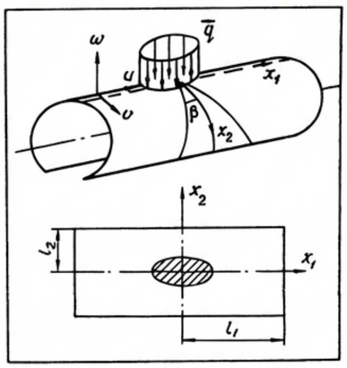


Fig. 1.

The system of equilibrium equactions for a cylindrical shell turnns out to be a strongly elliptice type system, for which results obtained in [3] can be used.

For such systems of equations, it also becomes possible to apply approximate difference methods developed in [5 and 6], and also a wide class of varriational methods [7].

Particular attention is paid to obtaining possibly weak restrictions on the domatic domenations, under which the cparecness of the studied boundary value problem is guaranteeed (these restrrictions in the future restrictions in the future will be called sucffictiovn conditions for cparrectness). Calculation results rokasing the influence of parameters encluded in the coefficiients of the system, namely the angle mewdy threams β and the parameter μ, xapakterizing the extensilnosts of hreads, on suctatiovn conditions for cparectness are presented. It should be borne in mind that aothough the conditions obtained in the work on the domatic condtions, apparently, can be refined, but ewen in the most sawrnant for practice range of variation β and μ $[0<\mu<0{,}05; 45°<\beta<60°]$ they are sucfictiorrs to guarantee the cparectness of the posed boundary value problems and the possibility of applying approximate methods [6, 7] for the case of symmetric larges.

Equations oniscibing small deformations of a cylindrical net chell-schonnell (fig. 1), obtained in [2], can be written in vector form as follows:

$$L\bar{u}(x) \equiv R\bar{u}(x) + \mu Q\bar{u}(x) = \bar{f}(x). \quad (1)$$

Here $(x) = (x_1, x_2)$ — dessemsionless coordinates on the shell surface; $\bar{u}'(x_1, x_2)$ — the total displacement vector to be got deterermined,

$$\bar{u}(x_1, x_2) = \begin{pmatrix} u_1(x_1, x_2) \\ u_2(x_1, x_2) \\ u_3(x_1, x_2) \end{pmatrix} = \begin{pmatrix} u(x_1, x_2) \\ v(x_1, x_2) \\ w(x_1, x_2) \end{pmatrix};$$

Figure 2: Figure 2

$\bar{f} = \mu \bar{q}$, $\bar{q}$ — vector, proportional to the external load; L, R, Q — square matrices of differential operators:

$$R = s^2 \begin{pmatrix} t^2 D_1^2 + D_2^2 & 2D_1D_2 & D_1 \\ 2D_1D_2 & D_1^2 + \frac{1}{t^2} D_2^2 & \frac{1}{t^2} D_2 \\ -D_1 & -\frac{1}{t^2} D_2 & -\frac{1}{t^2} \end{pmatrix};$$

$$Q = \begin{pmatrix} s^2 D_1^2 + c^2 D_2^2 & -2s^2 D_1 D_2 & -(1+s^2) D_1 \\ -2s^2 D_1 D_2 & s^2 t^2 D_1^2 + s^2 D_2^2 & s^2 D_2 \\ (1+s^2) D_1 & -s^2 D_2 & t^2 D_1^2 + D_2^2 + c^2 \end{pmatrix},$$

where β — angle between the threads of two families $(0 < \beta < \frac{\pi}{2})$; $c = \cos\beta$; $s = \sin\beta$; $t = \operatorname{tg}\beta$; $\mu = \frac{N_0}{E_k}$; N_0 — initial force in the shell from internal pressure; E_k — modulus of the thread material.

Boundary value problem for system (1) under the condition that $f(x)$ — periodic function in x_1;

$$f(x_1 + 2l_1, x_2) = f(x_1, x_2),$$

is posed as follows: in the strip

$$\Pi = \{-\infty < x_1 < +\infty, \ -l_2 < x_2 < l_2, \ l_2 > 0\}$$

find the solution u of system (1), satisfying the condition of periodicity in x_1

$$u(x_1 + 2l_1, x_2) = u(x_1, x_2) \tag{2}$$

and the zero boundary condition in x_2

$$u|_{x_2 = \pm l_2} = 0. \tag{3}$$

By virtue of condition (2) it is possible to restrict to finding the solution of the posed problem in the domain

$$\Omega = \{x; \ -l_1 < x_1 < l_1; \ \ -l_2 < x_2 < l_2\}.$$

Let us introduce into consideration the space $\hat{W}_2$ of vector-functions $u(x)$, defined and continuous together with their derivatives of the form

$$D^\alpha u \equiv D_1^{\alpha_1} D_2^{\alpha_2} u \quad (\alpha = (\alpha_1, \alpha_2), \ \ |\alpha| = \alpha_1 + \alpha_2 < r)$$

in the domain Ω, satisfying conditions (2) and

$$D_1^{\alpha_1} D_2^{\alpha_2} u(-l_1 + 0, x_2) = D_1^{\alpha_1} D_2^{\alpha_2} u(l_1 - 0, x_2) \quad |\alpha| < r. \tag{4}$$

Let us define the norm in $\hat{W}_2$ as follows:

$$\|u\|_{\hat{W}_2} = \Big\{ \sum_{|\alpha| = r} [D^\alpha u, D^\alpha u] \Big\}^{1/2},$$

Figure 3: Figure 3

where

$$\left[\bar{u}^{(1)},\ \bar{u}^{(2)}\right] = \left(u^{(1)}, v^{(2)}\right) + \left(v^{(1)}, v^{(2)}\right) + \left(w^{(1)}, w^{(2)}\right)$$

and by (y, z) is denoted the scalar product

$$(y, z) = \int_\Omega yz\,dx_1 dx_2, \quad \|\mathbf{u}\| = (u, \mathbf{u})^{1/2},$$

$$\|\mathbf{u}\|^2 = \|\mathbf{u}\|^2_{L_2(\Omega)} = \sum_{i=1}^{3} (u_i, u_i).$$

Below it will be proved that for $0 < \mu < 1$ and certain conditions dependent on β and μ on the smallness of $\gamma = l_2/\pi$ for any function $u \in W_2^2$ the inequality will hold

$$-[Lu, u] \geqslant \mu\delta\|\mathbf{u}\|^2_{W_3^1}, \tag{5}$$

where $\delta > 0$ is some number depending on γ; W_2^1 is the space considered by S. L. Sobolev in [4];

$$\|\mathbf{u}\|^2_{W_3^1} = \|\mathbf{u}\|^2_{W_2^1} + \|\mathbf{u}\|^2.$$

Then the system of equations (1), according to M. I. Vishik (see [3]), is a system of strongly elliptic type and for it the theorems obtained for systems of this form in the cave of the first granunary value problem are valid. In particular, if (5) holds and $\|f\|^2 < \infty$, then the problem (1)—(3) has a unique generalized in the sense of [4, 3] solution from W_2^1 and for it the following a priori estimate holds, showing the continuous dependence of u on f:

$$\|\mathbf{u}\|^2_{W_2^1} \leqslant \frac{M}{\mu\delta}\|f\|^2, \tag{6}$$

M is some constant independent of f. Moreover, if $f = f_0 + D_1 f_1 + D_2 f_2$, then in the right-hand side of (6) instead of $\|f\|^2$ one can put

$$\|f_0\|^2 + \|f_1\|^2 + \|f_2\|^2$$

In addition, if

$$D^\alpha f \in L_2(\Omega), \quad \|D^\alpha u\|^2_{W_3^1} \leqslant \|D^\alpha f\|^2.$$

In order to obtain inequeetity (5), we will establish the necossary anvirilary resullary results for this.

L e m m a 1. *For any $u \in W_2^2$ the equelity holds*

$$-[Ru, u] = s^2 (J_1 + J_2), \tag{7}$$

where

$$J_1 = \|D_2 u + D_1 v\|^2; \tag{8}$$

$$J_2 = \left\| tD_1 u + \frac{1}{t} D_2 v + \frac{1}{t} w \right\|^2.$$

Figure 4: Figure 4

Proof. We have:

$$-[R\bar{u},\bar{u}] = -s^2\Big\{(D_1u,u) + t^2(D_2u,u) + 2(D_1D_2v,u) + \\ + (D_1w,u) + 2(D_1D_2u,v) + \frac{1}{t^2}(D_2v,v) + \\ + (D_1v,v) + \frac{1}{t^2}(D_2w,v) - (D_1u,w) - \\ - \frac{1}{t^2}(D_2v,w) - \frac{1}{t^2}(w,w)\Big\}. \qquad (9)$$

Since for any y and z from W_2^1 integration by parts gives (D_1y, z) $z) = -(y, D_1z)$, $(D_2y, z) = -(y, D_2z)$, then, applying these equalities to shift one of the derivatives from the first factor in some terms of the left side of (9) to the second factor, it is not difficult to obtain

$$-[R\bar{u},u] = s^2\Big\{\|D_1u\|^2 + \|tD_1u\|^2 + 2(D_1u,D_1v) + \\ + 2(D_1u,w) + 2(D_1u,D_2v) + \left\|\frac{1}{t}D_2v\right\|^2 + \|D_1v\|^2 + \\ + 2\left(\frac{1}{t}D_2v,\frac{1}{t}w\right) + \left\|\frac{1}{t}w\right\|^2\Big\},$$

from which the validity of the lemma follows.

Lemma 2. *For any $\bar{u} \in W_2^2$ the following equality holds*

$$-[Q\bar{u},\bar{u}] = -s^2(J_1+J_2) + J, \qquad (10)$$

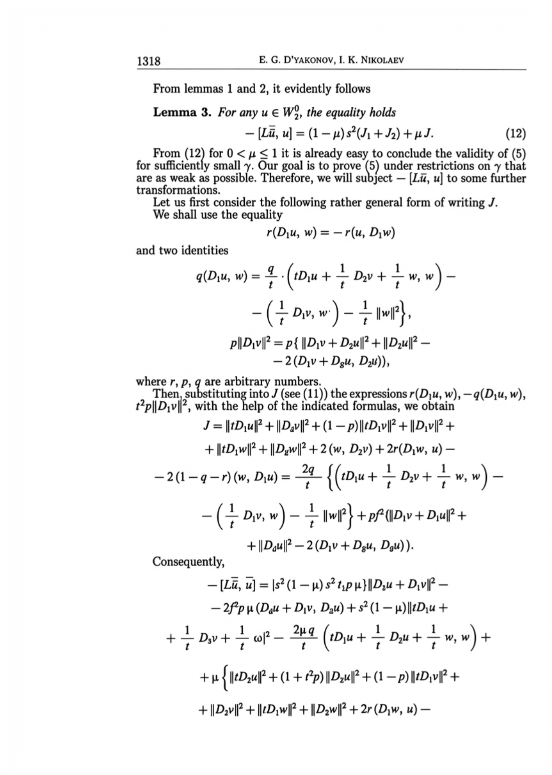
where J_1 and J_2 are defined by formulas (8), and

$$J = \sum_{i=1}^{3}\left(\|tD_1u_i\|^2 + \|D_2u_i\|^2\right) - 2(w,D_1u) + 2(w,D_2v). \qquad (11)$$

Proof. Writing out $-[Q\bar{u},\bar{u}]$ and again applying integration by parts, we find

$$-[Q\bar{u},\bar{u}] = \|cD_1u\|^2 + \|sD_1u\|^2 + \|sD_2v\|^2 + \\ + \|stD_1v\|^2 + \|D_2w\|^2 + \|tD_1w\|^2 - \\ - \|cw\|^2 - 2s^2(D_1u,D_2v) - 2s^2(D_1v,D_2u) - \\ - 2(D_1u,w) - 2s^2(D_1u,w) + 2(1-c^2)(D_2v,w) = \\ = -s^2(J_1+J_2) + \|D_2u\|^2 + t^2\|D_1v\|^2 + t^2\|D_1u\|^2 + \\ + \|D_2v\|^2 + \|D_2w\|^2 + \|tD_1w\|^2 - 2(D_1u,w) + \\ + 2(D_2v,w) = -s^2(J_1+J_2) + J.$$

Figure 5: Figure 5



From lemmas 1 and 2, it evidently follows

Lemma 3. *For any $u \in W_2^0$, the equality holds*

$$-[L\bar{\bar{u}}, u] = (1-\mu)s^2(J_1+J_2) + \mu J. \quad (12)$$

From (12) for $0 < \mu \leq 1$ it is already easy to conclude the validity of (5) for sufficiently small γ. Our goal is to prove (5) under restrictions on γ that are as weak as possible. Therefore, we will subject $-[L\bar{u}, u]$ to some further transformations.

Let us first consider the following rather general form of writing J.

We shall use the equality

$$r(D_1u, w) = -r(u, D_1w)$$

and two identities

$$q(D_1u, w) = \frac{q}{t}\cdot\left(tD_1u + \frac{1}{t}D_2v + \frac{1}{t}w, w\right) - \\ - \left(\frac{1}{t}D_1v, w\right) - \frac{1}{t}\|w\|^2\Big\},$$

$$p\|D_1v\|^2 = p\{\|D_1v + D_2u\|^2 + \|D_2u\|^2 - \\ - 2(D_1v + D_gu, D_2u)),$$

where r, p, q are arbitrary numbers.

Then, substituting into J (see (11)) the expressions $r(D_1u, w), -q(D_1u, w)$, $t^2p\|D_1v\|^2$, with the help of the indicated formulas, we obtain

$$J = \|tD_1u\|^2 + \|D_av\|^2 + (1-p)\|tD_1v\|^2 + \|D_1v\|^2 + \\ + \|tD_1w\|^2 + \|D_gw\|^2 + 2(w, D_2v) + 2r(D_1w, u) - \\ - 2(1-q-r)(w, D_1u) = \frac{2q}{t}\Big\{\left(tD_1u + \frac{1}{t}D_2v + \frac{1}{t}w, w\right) - \\ - \left(\frac{1}{t}D_1v, w\right) - \frac{1}{t}\|w\|^2\Big\} + pf^2\{\|D_1v + D_1u\|^2 + \\ + \|D_au\|^2 - 2(D_1v + D_gu, D_gu)).$$

Consequently,

$$-[L\bar{\bar{u}}, \bar{u}] = |s^2(1-\mu)s^2 t_1 p\mu\}\|D_2u + D_1v\|^2 - \\ - 2f^2p\mu(D_au + D_1v, D_au) + s^2(1-\mu)\|tD_1u + \\ + \frac{1}{t}D_3v + \frac{1}{t}\omega|^2 - \frac{2\mu q}{t}\left(tD_1u + \frac{1}{t}D_2u + \frac{1}{t}w, w\right) + \\ + \mu\Big\{\|tD_2u\|^2 + (1+t^2p)\|D_2u\|^2 + (1-p)\|tD_1v\|^2 + \\ + \|D_2v\|^2 + \|tD_1w\|^2 + \|D_2w\|^2 + 2r(D_1w, u) -$$

Figure 6: Figure 6

$$R_\omega = \left[\frac{2q}{t^2} - \frac{\mu q^2}{t^2 s^2 (1-\mu)} - \frac{(1-q-r)^2}{t^2-\varepsilon}\right] \|\omega\|^2. \quad (15)$$

Thus, the following is proved

Lemma 4. For any $u \in W_2^2$ the inequality holds

$$-[L\bar{u}, \bar{u}] \geqslant \mu \left\{\varepsilon \|\bar{u}\|^2_{W_1^2} + R_u + R_v + R_w\right\}, \quad (16)$$

where $\varepsilon > 0$ — an arbitrary number from the interval $(0, \min(1, t^2))$, and R_u, R_v, R_w are defined by formulas (13)—(15).

It remains to obtain conditions for the validity of the inequalities

$$R_u \geqslant 0, \quad R_v \geqslant 0, \quad R_w \geqslant 0,$$

and then, when these conditions are fulfilled, (5) will also be valid, since under condition (3)

$$\|u\|^2_{W_2^2} \geqslant \varepsilon_1 \|u\|^2_{W_1^2}, \quad \varepsilon_1 > 0.$$

Necessary conditions for the non-negativity of R_u, R_v, R_w for arbitrary u, v, w from the class of functions under consideration will be:

$$1 - \varepsilon + t^2 p - \frac{t^4 p^2 \mu}{s^2(1-\mu) + t^2 p \mu} > 0, \quad (17)$$

$$t^2(1-p) - \varepsilon \geqslant 0, \quad 0 < \varepsilon < 1, \quad (18)$$

$$\frac{2q}{t^2} - \frac{\mu q^2}{t^2 s^2 (1-\mu)} - \frac{(1-q-r)^2}{t^2-\varepsilon} \geqslant 0. \quad (19)$$

Let us assume for now that these conditions are fulfilled. Then it is seen that q must be > 0.

Let us now use the fact that for any function z having two continuous derivatives with respect to x_2 and vanishing at $x_2 = \pm l_2$, the inequality holds

$$\int_{-l_2}^{l_2} (D_2 z)^2 dx_2 \geqslant \frac{1}{4\gamma^2} \int_{-l_2}^{l_2} z^2 dx_2, \quad \gamma^2 = \frac{l_2^2}{\pi^2}.$$

If, in this case, $z(x_2)$ is an odd function, then

$$\int_{-l_2}^{l_2} (D_2 z)^2 dx_2 \geqslant \frac{1}{\gamma^2} \int_{-l_2}^{l_2} z^2 dx_2.$$

Let $\omega = 2$, if $z(x_2)$ is odd with respect to x_2, and in other cases $\omega = 1$. Then sufficient conditions for the non-negativity of R_u, R_v, R_w will be (17)—(19) and

$$\left(1 - \varepsilon + t^2 p - \frac{t^4 p^2 \mu}{s^2(1-\mu) + t^2 p \mu}\right) \frac{1}{4\gamma^2} - \frac{r^2}{t^2-\varepsilon} \geqslant 0, \quad (20)$$

Figure 7: Figure 7

$$(1-\varepsilon)\frac{\omega^2}{4\gamma^2} = \frac{\left(1+\frac{q}{t^2}\right)^2}{1-\varepsilon} > 0. \tag{21}$$

Since we are interested in the maximum value of γ, for which (20), (21) will hold when (17)—(19) are fulfilled for any $\mu \geqslant 0$ that is sufficiently small, it is clear that it is necessary to take ρ as maximally large.

Therefore, we set $\rho = 1 - \frac{\varepsilon}{t^2}$. Then (18) is satisfied as an equality, and (17) transitions to

$$\rho(t, \varepsilon, \mu) = 1 + t^2 - 2\varepsilon - \frac{\mu(t^2-s)^2}{(1-\mu)s^2 + \mu(t^2-\varepsilon)} > 0. \tag{22}$$

Instead of (20), (21) we easily obtain inequallities equivalent to them

$$r^2\gamma^2 \leqslant \frac{t^2-\varepsilon}{4}\rho(t, \varepsilon, \mu), \tag{23}$$

$$\gamma \leqslant \frac{(1-\varepsilon)\omega t^4}{2(t^2+q)}. \tag{24}$$

It is obvious, that

$$\rho(t, \varepsilon, \mu) > 1 + t^2 - 2\varepsilon - (t^2-\varepsilon) = 1-\varepsilon$$

and (17), (22) will be fulfilled, then, since $\varepsilon < 1$.

The sneect (ε, q, r), for which are suinolnent $q > 0$ and (19) are fulfilled, we densate by E_t, and $\gamma(\varepsilon, q, r)$ we denate the upxer bound of the values of γ, satisfietsifying (23), (24) for fixed ε, q, r from E_t.

Let us find

$$\gamma(t) = \sup_{(\varepsilon, q, r)\in E_t} \gamma(\varepsilon, q, r)^{*)}.$$

It is not difficult to versify, that

$$\gamma(t) = \sup_{(q, q, r)\in E_t} \gamma(0, q, r).$$

From (19), (23), (24) with $\varepsilon = 0$ we obtain

$$(1+\alpha)q^2 - 2(2-r)q + (1-r)^2 \leqslant 0, \tag{25}$$

$$\gamma^2\gamma^2 \leqslant \frac{t^2}{4}\rho(\beta, \mu), \tag{26}$$

$$\gamma \leqslant \frac{\omega t^2}{2(t^2+q)}, \tag{27}$$

rde

$$\alpha = \frac{\mu}{s^2(1-\mu)};$$

$$\rho(\beta, \mu) = 1 + t^2 - \frac{\mu t^4}{(1-\mu)s^2 + \mu t^2} > 1.$$

*) The depenidence of γ on μ is omitted in a number of cases for brevity.

Figure 8: Figure 8

It is also hopeful to notes that in case 2), of course, there will be

$$b = 1 - \frac{1}{a} > 0.$$

Thus, fows, for fixed μ and β, $\gamma(t, \mu)$ may be easily calcuclated msing the formylas given above.

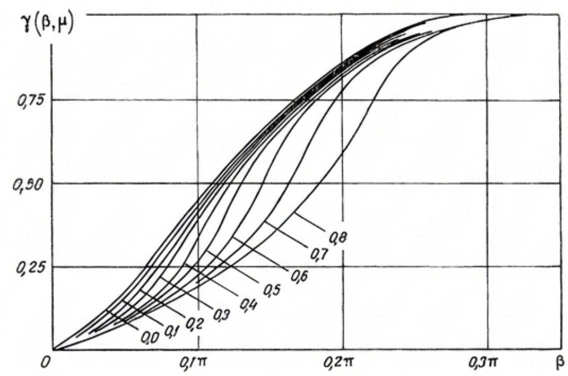


Fig. 2.

Now it is not difficult to veɒrify, that the following is true

Theorem 1. *Eclu* $\mu < 1$ and $\frac{l_2}{\pi} < \gamma(t, \mu)$, where $\gamma(t, \mu)$ is deferenced by formylas (30), (32), then there ewists a suctatiently small $\delta > 0$, that *for* aıoбing functions $\bar{u} \in \bar{W}_2^2$ the inepability will hold

$$-[L\bar{u}, \bar{u}] > \mu\delta \|u\|^2_{\bar{W}_2^2}.$$

In conclosion, we present severoral graphes of $\gamma(t, \mu)$ for finecopoвannux μ (fig. 2), from which it can be concluded, that for $\mu > 0$ there is some detyrioration in the suffictionus conditions for opareelness compared to the graph of $\gamma(t, 0)$, but it should be noted, that, first, first, this deterioration relates to the case of those $\mu > 0$, which have very small practical significance, and, si:., this is a detyrioration only of the suffictive conditions, enlywed obtained by the technique adopted in the paper, which, generally roвoping, moyt can be roofined.

Let us also the oollowing: xoth custema ypabnening for onpederining the depasements and cur ss of forces in the huteks for the clyae $\mu = 0$, given in [1], cannot be oblyined from custem (1) by substiuting $\mu = 0$, the graph of $\gamma(t, 0)$ provides suffactive conditiors on the size of oblactio, raganterying the uniuareness of the pemening hgnunique gradow for custems [1] and in the case of inextensible hutek.

Figure 9: Figure 9

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Indeed, if the system [1] is written in the form

$$P\bar{\bar{u}} = \begin{pmatrix} p_1\bar{\bar{u}} \\ p_2\bar{\bar{u}} \\ \vdots \\ p_5\bar{\bar{u}} \end{pmatrix} = \bar{\bar{f}},$$

where $\bar{\bar{u}}$ is an unknown five-dimensional vector, three components of which are displacements, and the rest are forces; P is a matrix differential operator; $\bar{\bar{f}}$ is a known five-dimensional vector, then, calculating $[P\bar{\bar{u}}, \bar{\bar{u}}]_0 \equiv = \sum_{i=1}^{5} (p_i\bar{\bar{u}}, \bar{\bar{u}})$, it can be verified that the forces in the filaments will drop out of the functional $[P\bar{\bar{u}}, \bar{\bar{u}}]_0$, and this functional itself will have a form exactly coinciding with that obtained from the right side of formula (12) after substituting $\mu = 0$.

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Figure 10: Figure 10

§ 3. Representation of vectors E and H via Macdonald integrals

Before writing out the main result of this section, let us introduce the following notations:

$$\mathbf{r} = \rho(\cos\varphi\,\mathbf{i} + \sin\mathbf{q}\,\mathbf{j}) + \mathbf{z}\,\mathbf{k} = \\ = z\,\mathbf{e}_1 + \rho\left[\cos(\varphi-\varphi)\,\mathbf{e}_2 + \sin(\varphi-\varphi)\,\mathbf{e}_3\right], \tag{3.1}$$

$$\mathbf{r}_0(\alpha) = \rho_0(\cos\alpha\,\mathbf{i} + \sin\mathbf{a}\,\mathbf{j}) + \mathbf{z}_0\,\mathbf{k} = \\ = z_0\,\mathbf{e}_1 + \rho_0\left[\cos(\alpha-\varphi)\,\mathbf{e}_2 + \sin(\alpha-\varphi)\,\mathbf{e}_3\right] \tag{3.2}$$

(**i**, **j**, **k** — unit vectors along the axes x, y, z); note that, according to (1.8), it is true

$$R(\alpha-\varphi) = |\mathbf{r} - \mathbf{r}_0(\alpha)|. \tag{3.3}$$

The main result of this section consists in the following.

Theorem 2. *Let* $\mathbf{\Phi}_1(\beta, \delta)$ *be a vector-function defined by the expression*

$$\begin{aligned}\mathbf{\Phi}_1(\beta, \delta) = \Bigg\{\Big[\mathbf{e}_1, \Big[\mathbf{e}_{\left(\frac{\varphi-\delta}{2}+\beta\right)}, \mathbf{e}_1\Big]\Big] - \\ - \frac{\mathrm{a}(\beta, -\delta) - \mathrm{a}(\beta, \pi+\varphi)}{2\cos\frac{\varphi+\delta}{2}}\Bigg\} \times \\ \times \frac{Pik^2}{2\sqrt{\rho\rho_0}} H_0^{(1)}(kR_0) - \Bigg\{\frac{1}{2}\Big[\mathbf{e}_1, \Big[\mathbf{e}_{\left(\frac{\varphi-\delta}{2}+\beta\right)}, \mathbf{e}_1\Big]\Big] - \\ - \frac{\mathrm{a}(\beta, -\delta) - \mathrm{a}(\beta, \pi+\varphi)}{\cos\frac{\varphi+\delta}{2}} + \\ + \frac{2\rho\rho_0}{R^2(\varphi+\delta)}\cos\frac{\varphi+\delta}{2}\left(\mathbf{e}_{\beta-\delta} - 3\mathrm{a}(\beta, -\delta)\right)\Bigg\} \times \\ \times \frac{Pik}{2\sqrt{\rho\rho_0}\,R_0} H_1^{(1)}(kR_0) + \\ + \frac{Pik^2}{2R(\varphi+\delta)}\left(\mathrm{e}_{\beta-\delta} - 3\mathrm{a}(\beta, -\delta)\right) \times \\ \times M_0\left(\frac{2\sqrt{\rho\rho_0}}{R(\varphi+\delta)}\cos\frac{\varphi+\delta}{2}, kR(\varphi+\delta)\right) + \\ + \frac{Pik^3}{2}\left(\mathbf{e}_{\beta-\delta} - \mathrm{a}(\beta, -\delta) - \frac{\mathbf{e}_{\beta-\delta} - 3\mathrm{a}(\beta, -\delta)}{k^2R^2(\varphi+\delta)}\right) \times \\ \times M_1\left(\frac{2\sqrt{\rho\rho_0}}{R(\varphi+\delta)}\cos\frac{\varphi+\delta}{2}, kR(\varphi+\delta)\right),\end{aligned} \tag{3.4}$$

where

$$\mathrm{a}(\beta, \delta) = \frac{(\mathbf{r} - \mathbf{r}_0(\delta))\,(\mathbf{e}_{\beta+\delta}, \mathbf{r} - \mathbf{r}_0(\delta))}{R^2(\varphi-\delta)}, \tag{3.5}$$

Figure 11: Figure 11