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Abstract

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MATHEMATICS

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A NON-SELF-ADJOINT DIFFERENCE OPERATOR

(Presented by Academician I. N. Vekua on 10 VI 1966)

In the present note a difference analogue of the non-self-adjoint Sturm—Liouville operator on the half-line is studied.

1. We shall consider the difference expression l, which transforms a sequence of complex numbers $y = (y_0, y_1, y_2, ...)$ into the sequence $ly = ((ly)_1, (ly)_2, ...)$ by the formulas

$$(ly)_j = \frac{1}{2}(y_{j-1} + y_{j+1}) - b_{jy}j, \qquad j = 1, 2, ... ; \tag{1}$$

here $(b_1, b_2, ...)$ is a given sequence of complex numbers. We shall be interested in certain solutions y of the difference equation

$$(ly)_j = \lambda y_j = 0 \qquad (\lambda = \tfrac{1}{2}(\rho^{-1} + \rho),\ |\rho| \geqslant 1), \tag{2}$$

containing the complex parameter λ, which it is convenient to regard as the function of ρ indicated in (2).

2. Let $s_j(\lambda)$, $c_j(\lambda)$ be solutions of equation (2) satisfying the initial conditions $s_0(\lambda) = 0$, $s_1(\lambda) = 1$, $c_0(\lambda) = 1$, $c_1(\lambda) = 0$*. These solutions are polynomials in λ of degrees $j-1$ and $j-2$, respectively. For each $\delta > 0$ denote by W_δ the following domain in the complex λ-plane: $W_\delta = \{\rho :\ |\rho| \geqslant 1,\ |\rho^2 - 1| \geqslant \delta\}$. There exists a $C > 0$, and for each $\delta > 0$ there exists a $C_\delta > 0$, such that for $\rho \in W_\delta$ and for all $j = 2, 3, ...$

$$|s_j(\lambda)| \leqslant C|\rho|^{j-1} \exp C_\delta |\rho|^{-1} \sum_{l=1}^{j-1} |b_l|, \qquad |c_j(\lambda)| \leqslant C|\rho|^{j-2} \exp C_\delta |\rho|^{-1} \sum_{l=1}^{j-1} |b_l|.$$

3. Put

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$$\sigma_j = \sum_{l=j+1}^{\infty} |b_l|, \qquad \sigma_{1j} = \sum_{l=j+1}^{\infty} l|b_l|.$$

If $\sigma_{1j} < \infty$, then equation (2) has a solution $e_j(\rho)$, which can be represented in the form

$$e_j(\rho) = \rho^{-j} + \sum_{l=j+1}^{\infty} k_{jl}\rho^{-l}, \qquad |\rho| \geqslant 1, \qquad j = 0, 1, 2, \dots,$$

and moreover

$$|k_{jl}| \leqslant Ce^{\sigma_{1j}}\sigma_{[(j+l)/2]}, \qquad l > j = 0, 1, 2, \dots;$$

here C is a certain number, and $[x]$ is the integer part of x. The assertion just formulated is analogous to the well-known theorem on the transformation operator for solutions of differential equations.

It is easy to see that the solution $e_j(\rho)$ is continuous in ρ for $|\rho| \geqslant 1$ and holomorphic for $|\rho| > 1$, $j = 0, 1, 2, \dots$.

For the existence of a solution $e_j(\rho)$ equal to $\rho^{-j}(1 + o(1))$ as $j \to \infty$, it is sufficient that $\sigma_j < \infty$.

4. If $\sigma_j < \infty$, then for each $\delta > 0$ there exists a natural number h_δ such that equation (2) has a solution $\hat{e}_j(\rho)$, satisfying asymptoti-

* If $b_1 = b_2 = \dots = 0$, then $s_j(\lambda) = (\rho^{-j} - \rho^j)(\rho^{-1} - \rho)^{-1}$ and is a difference analogue of the function $\lambda^{-1/2}\sin x\lambda^{1/2}$; $c_j(\lambda)$ is a difference analogue of $\cos x\lambda^{1/2}$.

to the asymptotic equality

$$\hat{e}_j(\rho) = \rho^j[1 + O(1/\rho)], \qquad |\rho| \to \infty$$

uniformly for $j > h_\delta$, $\rho \in W_\delta$. Moreover, for every $\alpha > 0$

$$\hat{e}_j(\rho) = \rho^{(j)}[1 + o(1)], \qquad j \to \infty,$$

uniformly with respect to ρ for $\rho \in W_\delta$, $|\rho| > 1 + \alpha$.

If, for some $\gamma > 0$, the series $\sum l^{1+\gamma}|b_l|$ converges, then for $\lambda = 1$ and $(\lambda = -1)$ equation (2) has a solution $\hat{e}_j(\rho)$ for which

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$$\hat{e}_j(1) = j[1 + O(j^{-\gamma})], \qquad j \to \infty$$

$$(\hat{e}_j(-1) = (-1)^j j[1 + O(j^{-\gamma})], \qquad j \to \infty).$$

5. The Wronskian $w(\alpha_j, \beta_j) = \alpha_j \beta_{j+1} - \alpha_{j+1}\beta_j$ of any pair of solutions α_j, β_j of equation (2) does not depend on j. We have

$$w(e_j(\rho), \hat{e}_j(\rho)) = \rho - \rho^{-1}, \qquad \rho \in W_\delta.$$

6. Denote by $H = l_2[1, \infty)$ the Hilbert space of sequences $y = (y_1, y_2, ...)$ with norm

$$\|y\| = \left(\sum_{n=1}^{\infty} |y_n|^2 \right)^{1/2}.$$

For each sequence $y \in H$ set

$$y_0 = \theta y_1, \tag{3}$$

where θ is a fixed complex number. In what follows, when computing $(ly)_1$, we shall take into account the "boundary" condition (3), and by $Ly = L_\theta y$ we shall denote the sequence $((ly)_1, (ly)_2, ...)$. If $\sup |b_j| < \infty$, in particular if $\sigma_j < \infty$, then L is a continuous linear operator mapping all of H into itself.

7. Let $\sigma_j < \infty$. Then the operator L has no eigenvalues λ in the interval $(-1, +1)$. The spectrum of the operator L consists of the segment $[-1, +1]$ and eigenvalues λ determined by the formula $\lambda = \frac{1}{2}(\rho^{-1} + \rho)$, where ρ is a root of the equation $\theta e_1(\rho) - e_0(\rho) = 0$ and $|\rho| > 1$. The eigenvalues of L form a bounded (at most countable) set, whose limit points may lie only on the segment $[-1, +1]$. All points $\lambda \in (-1, +1)$ belong to the continuous spectrum of L. If the series $\sum l^{1+\gamma}|b_l|$ converges for some $\gamma > 0$, then the points $\lambda = \pm 1$ also belong to the continuous spectrum of L.

8. Let $\sigma_j < \infty$ and let λ be a point of the resolvent set, and $R_\lambda = (L - \lambda)^{-1}$ the resolvent of the operator L. Put

$$a(\rho) = \theta e_1(\rho) - e_0(\rho), \qquad \omega_j(\lambda) = s_j(\lambda) + \theta c_j(\lambda);$$

$$R_{jl}(\lambda) = \begin{cases} 2e_j(\rho)\omega_l(\rho)/a(\rho), & \text{for } l = 1, \dots, j-1, \\ 2\omega_j(\rho)e_l(\rho)/a(\rho), & \text{for } l = j, j+1, \dots . \end{cases}$$

Then for all $f \in H$

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$$(R_\lambda f)_j = \sum_{l=1}^{\infty} R_{jl}(\lambda) f_l, \qquad j = 1, 2, \dots .$$

There exists $C > 0$, and for every $\delta > 0$ there exists C_δ such that

$$C/|a(\rho)|\sqrt{|\rho| - 1} \le \|R_\lambda\| \le C_\delta/|a(\rho)|(|\rho| - 1),$$

where the second inequality holds only for $\rho \in W_\delta$.

9. Everywhere in what follows we assume that, for some $\varepsilon > 0$,

$$\sum_{j=1}^{\infty} (1+\varepsilon)^j |b_j| < \infty. \tag{4}$$

Only under this assumption (which is analogous to the condition of exponential decrease of the "potential," introduced in the case of a differential operator by M. A. Naimark) shall we construct spectral expansions corresponding to the operator L.

Now the solution $e_j(\rho)$ (see Sec. 3) admits an analytic continuation from the domain $|\rho| > 1$ to a function holomorphic in the domain $|\rho| > (1+\varepsilon)^{-1/2}$. Therefore the equation $\theta e_1(\rho) - e_0(\rho) = 0$ has only a finite number of solutions in the domain $|\rho| \ge 1$, and, in particular, the operator L has only a finite number of eigenvalues. The roots of the equation $\theta e_1(\rho) - e(\rho) = 0$ such that $|\rho| \ge 1$ will be called the singular numbers of the operator L. Denote by $\rho_1, \dots, \rho_\alpha$ the singular numbers for which $|\rho_k| > 1$, $k = 1, \dots, \alpha$, and by $\rho_{\alpha+1}, \dots, \rho_\beta$ those singular numbers for which $|\rho_k| = 1$, $k = \alpha+1, \dots, \beta$. Let $\lambda_k = \frac{1}{2}(\rho_k^{-1} + \rho_k)$. For $k = 1, \dots, \alpha$, the number λ_k is an eigenvalue of the operator (and there are no other eigenvalues). The numbers $\lambda_{\alpha+1}, \dots, \lambda_\beta$ belong to the continuous spectrum; we shall call them spectral singularities of the operator L.

In what follows, by m_k we denote the multiplicity of the root ρ_k of the equation $\theta e_1(\rho) - e_0(\rho) = 0$. The numbers $m_1, \dots, m_\alpha$ coincide with the multiplicities of the eigenvalues $\lambda_1, \dots, \lambda_\alpha$, i.e., with the dimensions of the corresponding root subspaces. The numbers $m_{\alpha+1}, \dots, m_\beta$ will be called the multiplicities of the spectral singularities $\lambda_{\alpha+1}, \dots, \lambda_\beta$.

10. Denote by $\mathcal{B}_{kj}(\lambda)$ any bounded measurable functions on the interval $-1 \le \lambda \le +1$, holomorphic in a neighborhood of the spectral singularities $\lambda_{\alpha+1}, \dots, \lambda_\beta$ and satisfying the conditions

$$\left\{ \left[\left(\frac{d}{d\lambda}\right)^{j'} \mathcal{B}_{kj}(\lambda) \right]_{\lambda=\lambda_{k'}} \right\} = \begin{cases} 1, & \text{if } j = j',\ k = k', \\ 0, & \text{in all other cases.} \end{cases}$$

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For an arbitrary function $\Phi(\lambda)$, differentiable m_k-1 times at the point λ_k, $k=\alpha+1,\dots,\beta$, put

$$[\mathcal{B}\Phi(\lambda)] = \Phi(\lambda) - \sum_{k=\alpha+1}^{\beta}\sum_{j=0}^{m_k-1} \mathcal{B}_{kj}(\lambda)\Phi^{(j)}(\lambda_k).$$

The point λ_k is a root of the function $[\mathcal{B}\Phi(\lambda)]$ of multiplicity at least m_k, $k=\alpha+1,\dots,\beta$. The following expansion in eigenfunctions of the operator L for the "kernel" of the resolvent (see Sec. 8) is valid:

$$R_{jl}(z) = \frac{2}{\pi}\int_{-1}^{+1}\left[\mathcal{B}\frac{\omega_j(\lambda)\omega_l(l)}{\lambda-z}\right]\frac{i\sqrt{1-\lambda^2}\,d\lambda}{a(\lambda+i\sqrt{1-\lambda^2})\,a(\lambda-i\sqrt{1-\lambda^2})}+$$
$$+\sum_{k=1}^{\beta}\left\{\left[\left(\frac{d}{d\lambda}\right)^{m_k-1} M_k(\lambda)\frac{\omega_j(\lambda)\omega_l(\lambda)}{\lambda-z}\right]_{\lambda=\lambda_k}\right\}; \qquad (5)$$

here $\sqrt{1-\lambda^2}\geq 0$ for $\lambda\in[-1,+1]$, $a(\rho)=\theta e_1(\rho)-e_0(\rho)$, and the functions $M_k(\lambda)$ for $k=\alpha+1,\dots,\beta$ depend on the choice of the functions $\mathcal{B}_{kj}(\lambda)$. The subintegral function in formula (5) is bounded. If there are no spectral singularities, then the operator $\mathcal{B}$ in formula (5) is superfluous.

11. Introduce the notation $\omega_f(\lambda)=\sum_{j=1}^{\infty} f_j\omega_j(\lambda)$. There exists a number $C>0$ such that

$$\int_{-1}^{+1}|\omega_f(\lambda)|^2\sqrt{1-\lambda^2}\,d\lambda \leq C\sum_{j=1}^{\infty}|f_j|^2.$$

Let $\omega_j^{(m)}(\lambda)=(d/d\lambda)^m\omega_j(\lambda)$; then

$$w_f^{(m)}(\lambda_k)=\sum_{j=1}^{\infty} f_j\omega_j^{(m)}(\lambda_k),$$

$k=1,\dots,\alpha;\ m=0,\dots,m_k-1$, are continuous functionals of $f\in H$. The collection of quantities $\omega_f(\lambda)$, $\lambda\in[-1,+1]$, $\omega_f^{(m)}(\lambda_k)$, $k=1,\dots,\alpha$, $m=0,\dots,m_k-1$, will be called the L-Fourier transform of the element $f\in H$.

There exist linear continuous functionals $M_{km}: f\to M_{km}(f)$, defined on the space H, such that for every element $f\in H$ the following expansion in eigenfunctions of the operator L is valid:

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$$f_j = \frac{2}{\pi} \int_{-1}^{+1} [\mathfrak{B}\omega_j(\lambda)]\, \omega_f(\lambda) \frac{\sqrt{1-\lambda^2}\, d\lambda}{a(\lambda + i\sqrt{1-\lambda^2})\, a(\lambda - i\sqrt{1-\lambda^2})} + \sum_{k=1}^{\beta} \sum_{m=0}^{m_k - 1} M_{km}(f)\, \omega_j^{(m)}(\lambda_k). \tag{6}$$

For $k = 1, \dots, \alpha$ the formulas for the functionals M_{km} are not difficult to write explicitly. For $k = \alpha + 1, \dots, \beta$ the values of these functionals are uniquely determined by the condition $f \in H = l_2[1, \infty)$. Formula (6) may be interpreted as the inversion formula for the L-Fourier transform of a function $f \in H$. Using methods analogous to those employed in the case of differential operators, the L-Fourier transform can be extended to functions f_j growing arbitrarily as $j \to \infty$. The extended L-Fourier transform is not uniquely invertible: it is zero on the subspace spanned by the principal functions of the spectral singularities, i.e., on the functions $\omega_j^{(m)}(\lambda_k)$, $k = \alpha + 1, \dots, \beta;\ m = 0, \dots, m_k - 1$.

12. Denote by $\mathfrak{S}$ the manifold of those functions $f \in H$ for which

$$\int_{-1}^{+1} \left| \frac{\omega_f(\lambda)}{a(\lambda + i\sqrt{1-\lambda^2})} \right|^2 \sqrt{1-\lambda^2}\, d\lambda < \infty.$$

The manifold $\mathfrak{S}$ is dense in the space H. For any $f \in \mathfrak{S}$ and $g \in H$, the following generalized Parseval equality holds:

$$\sum_{j=1}^{\infty} f_j g_j = \frac{2}{\pi} \int_{-1}^{+1} \omega_f(\lambda) \omega_g(\lambda) \frac{\sqrt{1-\lambda^2}\, d\lambda}{a(\lambda + i\sqrt{1-\lambda^2})\, a(\lambda - i\sqrt{1-\lambda^2})} + \sum_{k=1}^{\alpha} \left\{ \left(\frac{d}{d\lambda}\right)^{m_k - 1} M_k(\lambda) \omega_f(\lambda) \omega_g(\lambda) \right\}_{\lambda = \lambda_k}.$$

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Note: Figure translations are in progress. See original paper for figures.

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