ON ELECTROMAGNETIC POTENTIALS IN GYROTROPIC MEDIA

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Abstract

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ON ELECTROMAGNETIC POTENTIALS IN GYROTROPIC MEDIA

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The article gives a method for solving boundary-value problems of the electrodynamics of gyrotropic media, which is a generalization of the method for solving problems for anisotropic media with the aid of potentials, proposed by A. N. Tikhonov ([1]) and developed for uniaxial media in ([2]).

For definiteness, we consider the case of a plasma in a magnetic field, whose tensor of relative electric permittivity has the form ([3])

$$\hat{\varepsilon} = \left\{ \begin{array}{ccc} \varepsilon & i\varepsilon_a & 0 \\ -i\varepsilon_a & \varepsilon & 0 \\ 0 & 0 & \varepsilon_z \end{array} \right\}. \tag{1}$$

We shall assume the plasma filling some region of space to be homogeneous and, as usual, quasineutral and nonmagnetic, and the electromagnetic field to depend on time according to the harmonic law $\exp i\omega t$. The results can be transferred directly to the case of a polycrystalline ferrite.

1. With the aid of electrodynamic potentials defining the field,

$$\mathbf{H} = \operatorname{rot} \mathbf{A}, \quad \mathbf{E} = -ik_0\mathbf{A} - \operatorname{grad} \Phi \quad (k_0 = \omega/c), \tag{2}$$

the solution of Maxwell’ s equations is reduced to satisfying the system

$$\Delta \mathbf{A} - \operatorname{grad} \operatorname{div} \mathbf{A} - ik_0\hat{\varepsilon} \operatorname{grad} \Phi + k_0^2 \hat{\varepsilon} \mathbf{A} = 0, \tag{3}$$

$$\operatorname{div} \hat{\varepsilon} \operatorname{grad} \Phi + ik_0 \operatorname{div} \hat{\varepsilon} \mathbf{A} = 0. \tag{4}$$

This system, when written out in Cartesian coordinates, is a linear system of differential equations with constant coefficients and can be investigated by the general method of elimination of variables ([4]). The determinant formed from the operator coefficients of this system proves to be identically equal to zero.

This means that the equations of the system are dependent and, in particular, equation (4) is satisfied, under the condition of sufficient smoothness of the functions, as a consequence of the satisfaction of the three scalar equations (3).

It follows from the latter equations that among the 4 unknown functions there exists a dependence which may be conditionally expressed, by the known relations of linear algebra in terms of the minors of the matrix of operator coefficients, as follows:

$$A_x : A_y : A_z : \Phi = \frac{\partial}{\partial x} L_4 : \frac{\partial}{\partial y} L_4 : \frac{\partial}{\partial z} L_4 : -ik_0 L_4, \tag{5}$$

where L_4 is a linear differential operator of the 4th order,

$$\begin{aligned} L_4 = {} & \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right)^2 + \left(1 + \frac{\varepsilon_z}{\varepsilon}\right) \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \frac{\partial^2}{\partial z^2} + \frac{\varepsilon_z}{\varepsilon} \frac{\partial^4}{\partial z^4} \\ & + k_0^2 \left(\varepsilon_z + \frac{\varepsilon^2 - \varepsilon_a^2}{\varepsilon} \right) \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) + 2k_0^2 \varepsilon_z \frac{\partial^2}{\partial z^2} + k_0^4 \varepsilon_z \frac{\varepsilon^2 - \varepsilon_a^2}{\varepsilon}. \end{aligned} \tag{6}$$

The meaning of the conditional notation (5) is that

$$-ik_0 L_4 A_x = \frac{\partial}{\partial x} L_4 \Phi, \qquad -ik_0 L_4 A_y = \frac{\partial}{\partial y} L_4 \Phi, \qquad -ik_0 L_4 A_z = \frac{\partial}{\partial z} L_4 \Phi,$$

whence, taking into account expressions (2), it follows that the electromagnetic potentials that produce a nonzero electromagnetic field satisfy the equation

$$L_4 X = 0 \quad (X = A_x, A_y, A_z, \Phi). \tag{7}$$

As is known ([5]), this equation is satisfied by the components of the electric and magnetic fields in gyrotropic media; direct treatment of these components is equivalent to introducing vector potentials alone, with zero divergence, without scalar potentials. In solving problems directly in terms of the field components, difficulties arise connected with satisfying the boundary conditions and the conditions for excitation of the field by sources; these difficulties were the reason for introducing potentials also in the treatment of isotropic media.

The introduction of electrodynamic potentials with nonzero divergence for anisotropic media, in order to satisfy the boundary conditions regularly, is due to A. N. Tikhonov ([1]), who found a normalization relation for the potentials in place of the Lorentz condition, which proved in a certain sense to be optimal ([2]).

2. For the normalization of potentials in gyrotropic media we shall use the generalized Lorentz condition ([6])

$$\Phi = -\frac{1}{ik_0}\operatorname{div}\hat{e}\mathbf{A} \tag{8}$$

with the tensor $\hat{e}$ having the form

$$\hat{e} = \begin{Bmatrix} e & ie_a & 0 \\ -ie_a & e & 0 \\ 0 & 0 & e_z \end{Bmatrix} \tag{9}$$

for arbitrary constants e, e_a, and e_z.

Eliminating Φ from system (3) in this way, we obtain three equations for A_x, A_y, and A_z

$$\begin{gathered}
\left\{(\varepsilon^2-\varepsilon_a^2)e\frac{\partial^2}{\partial x^2} + [i\varepsilon_a - ie_a(\varepsilon^2-\varepsilon_a^2)]\frac{\partial^2}{\partial x\,\partial y} + \varepsilon\frac{\partial^2}{\partial y^2} + \varepsilon\frac{\partial^2}{\partial z^2} + k_0^2(\varepsilon^2-\varepsilon_a^2)\right\}A_x + \\
+\left\{[-i\varepsilon_a + ie_a(\varepsilon^2-\varepsilon_a^2)]\frac{\partial^2}{\partial x^2} + [(\varepsilon^2-\varepsilon_a^2)e-\varepsilon]\frac{\partial^2}{\partial x\,\partial y} - i\varepsilon_a\frac{\partial^2}{\partial z^2}\right\}A_y + \\
+\left\{[(\varepsilon^2-\varepsilon_a^2)e_z-\varepsilon]\frac{\partial^2}{\partial x\,\partial z} + i\varepsilon_a\frac{\partial^2}{\partial y\,\partial z}\right\}A_z = 0,
\end{gathered}$$

$$\begin{gathered}
\left\{[i\varepsilon_a - ie_a(\varepsilon^2-\varepsilon_a^2)]\frac{\partial^2}{\partial y^2} - [(\varepsilon^2-\varepsilon_a^2)e-\varepsilon]\frac{\partial^2}{\partial y\,\partial x} + i\varepsilon_a\frac{\partial^2}{\partial z^2}\right\}A_x + \\
+\left\{\varepsilon\frac{\partial^2}{\partial x^2} + [-i\varepsilon_a + ie_a(\varepsilon^2-\varepsilon_a^2)]\frac{\partial^2}{\partial y\,\partial x} + (\varepsilon^2-\varepsilon_a^2)e\frac{\partial^2}{\partial y^2} + \varepsilon\frac{\partial^2}{\partial z^2} + k_0^2(\varepsilon^2-\varepsilon_a^2)\right\}A_y + \\
+\left\{[(\varepsilon^2-\varepsilon_a^2)e_z-\varepsilon]\frac{\partial^2}{\partial y\,\partial z} - i\varepsilon_a\frac{\partial^2}{\partial x\,\partial z}\right\}A_z = 0,
\end{gathered}$$

$$\begin{gathered}
\left[(\varepsilon_z e-1)\frac{\partial^2}{\partial z\,\partial x} - ie_a\varepsilon_z\frac{\partial^2}{\partial z\,\partial y}\right]A_x + \left[ie_a\varepsilon_z\frac{\partial^2}{\partial z\,\partial x} + (\varepsilon_z e-1)\frac{\partial^2}{\partial z\,\partial y}\right]A_y + \\
+\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \varepsilon_z e_z\frac{\partial^2}{\partial z^2} + k_0^2\varepsilon_z\right]A_z = 0.
\end{gathered}$$

The determinant of the matrix of operator coefficients of this linear system with constant coefficients proves to be equal to $(e^2-e_a^2)L_4L_2$, where the second-order operator L_2 depends on the components of the tensor (9) and gives rise to the equation

$$L_2X \equiv (\operatorname{div}\hat{e}\operatorname{grad} + k_0^2)\,X = \left[e\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) + e_z\frac{\partial^2}{\partial z^2} + k_0^2\right]X = 0, \tag{10}$$

which the components of the potential may satisfy along with the basic equation (7).

3. The appearance of equation (10), which is superfluous from the point of view of the possibilities of representing the electromagnetic field, does not contradict relations (5). According to the latter, system (3)–(4) admits the existence of potentials that do not satisfy equation (7), but are necessarily connected by the condition

$$\mathbf{A}^* = -\frac{1}{ik_0}\,\mathrm{grad}\,\Phi^* \tag{11}$$

and therefore make no contribution to the electromagnetic field. In this case no restrictions are imposed on the function Φ^*.

As is readily seen [6], equation (10) is a consequence of imposing, on the gradient transformation

$$\mathbf{A}' = \mathbf{A} - \frac{1}{ik_0}\,\mathrm{grad}\,\Phi^*, \qquad \Phi' = \Phi + \Phi^*,$$

the additional condition of normalization of the potentials (8), which relates both $\mathbf{A}$ and Φ, and $\mathbf{A}'$ and Φ', and is a generalization of the known Stratton condition [7] for the possibility of uniquely determining the potentials associated with the usual Lorentz condition to the case of generalized conditions.

4. Introducing solutions of the superfluous equation into consideration makes it possible to satisfy the boundary conditions for the potentials in a regular manner and to determine them, for the chosen generalized Lorentz condition, uniquely.

In the case of an isotropic medium the superfluous equation corresponding to the ordinary Lorentz condition reduces to the principal one, since the latter degenerates into a second-order equation ($L_4 = L_2 \cdot L_2$), and each of the three components of the vector potential is its independent solution.

In anisotropic media the principal equation is of the 4th order and, in the number of independent solutions, corresponds to two equations of the 2nd order, although, except for the case of uniaxial media, it does not split into them. If the components of the vector potential were independent, there would be 6 independent solutions corresponding to solutions of second-order equations, instead of 3 for an isotropic medium. However, additional connections imposed by the system exist between the components of the vector potential. In solving problems through the fields, it has been proposed [5] to take into account additionally one of the equations of the system.

From pairwise consideration of the equations of our system there follow 2 additional connections between the components through algebraic complements of the elements of the matrix of operator coefficients,

$$A_x : A_y : A_z = A_{i1} : A_{i2} : A_{i3}, \tag{12}$$

where i is the number of the omitted equation. Calculations give, for example,

$$A_{31} = -i\varepsilon_a(\varepsilon^2 - \varepsilon_a^2)\frac{\partial^2}{\partial z\,\partial y}L_2 + (\varepsilon^2 - \varepsilon_a^2)\frac{\partial^2}{\partial z\,\partial x}M,$$

$$A_{32} = i\varepsilon_a(\varepsilon^2 - \varepsilon_a^2)\frac{\partial^2}{\partial z\,\partial x}L_2 + (\varepsilon^2 - \varepsilon_a^2)\frac{\partial^2}{\partial z\,\partial y}M,$$

$$A_{33} = \varepsilon(\varepsilon^2 - \varepsilon_a^2)\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} + k_0^2\frac{\varepsilon^2 - \varepsilon_a^2}{\varepsilon}\right)L_2 + (\varepsilon^2 - \varepsilon_a^2)\frac{\partial^2}{\partial z^2}M,$$

where

$$M = (1 - \varepsilon\varepsilon_z + \varepsilon_a e_a)\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) + (1 - \varepsilon\varepsilon_z)\frac{\partial^2}{\partial z^2} + k_0^2\{\varepsilon - (\varepsilon^2 - \varepsilon_a^2)\varepsilon_z\}.$$

As a result of imposing conditions (12) on each type of solution of the principal equation, of the aforementioned 6 solutions only 2 independent ones remain. The introduction of the superfluous equation gives the 3rd solution, which is necessary, generally speaking, for the possibility of regular satisfaction of the boundary conditions.

5. Let us note that from the last expressions, substituted into (12), there follows directly relation (11) between the components satisfying the superfluous equation, under the condition that the operator M does not turn···

identically equal to zero. This special case is realized for the following values of the components of tensor (9):

$$e = \frac{\varepsilon}{\varepsilon^2 - \varepsilon_a^2}, \qquad e_a = \frac{\varepsilon_a}{\varepsilon^2 - \varepsilon_a^2}, \qquad e_z = \frac{\varepsilon \pm \varepsilon_a}{\varepsilon^2 - \varepsilon_a^2}. \tag{13}$$

The special generalized Lorentz condition with tensor (13), in solving problems, affords a known advantage associated with lowering the order of the differential relations (12), which take the form

$$A_x = \pm iA_y,$$

$$\left[\varepsilon\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)+(\varepsilon\mp\varepsilon_a)\frac{\partial^2}{\partial z^2}+k_0^2(\varepsilon^2-\varepsilon_a^2)\right]A_x=-i\varepsilon_a\frac{\partial}{\partial z}\left(\frac{\partial}{\partial y}+i\frac{\partial}{\partial x}\right)A_z.$$

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Note: Figure translations are in progress. See original paper for figures.