A NEW FORMULATION OF QUANTUM THEORY

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Abstract

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A NEW FORMULATION OF QUANTUM THEORY

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Quantum field theory has usually been formulated in terms of creation and annihilation operators $\hat{a}^+(k)$, $\hat{a}(k)$, subject to canonical commutation or anticommutation relations and acting in a Hilbert space H_a containing the vacuum vector θ_a, i.e., a vector satisfying the conditions $\hat{a}(k)\theta_a = 0$ (Fock space). However, after the proof of Haag' s well-known theorem it became clear that the Hamiltonian formalism in the space H_a is manifestly incorrect in the case of relativistic quantum field theory. This stimulated the study of so-called pathological representations of the commutation relations (representations in which there is no vacuum vector), but the consideration of these representations has not led to the construction of a correct theory.

In the present note a representation of state vectors by functionals is constructed and studied, in a certain sense uniting state vectors belonging to inequivalent representations of the commutation relations. Using this functional representation of state vectors, one can indicate a new formulation of quantum theory that does not appeal to the concept of Hilbert space. This formulation apparently makes it possible, in a certain sense, to rehabilitate the Hamiltonian approach in quantum field theory. As an illustration, a model of a scalar field interacting with a source is considered. The introduced representation of state vectors also makes it possible to consider from a new point of view the question of locality in quantum field theory.

Thus, let $\hat{a}^+(k)$, $\hat{a}(k)$ be creation and annihilation operators (more precisely, operator-valued generalized functions) in a Hilbert space H, satisfying the relations $[\hat{a}(k), \hat{a}^+(l)] = \delta(k-l)$ for bosons and $[\hat{a}(k), \hat{a}^+(l)]_+ = \delta(k-l)$ for fermions (here k, l range over some set with a measure). To a normalized vector $\Phi \in H$ in the Bose case we associate the functional $L(\alpha^*, \alpha) = (e^{-\alpha\hat{a}^+} e^{\alpha^*\hat{a}}\Phi, \Phi)$ (here $\alpha^*(k)$, $\alpha(k)$ are complex conjugate numerical functions; expressions of the type $\alpha\hat{a}^+$ here and below are understood as $\int \alpha(k)\hat{a}^+(k)\,dk$). In the Fermi case we associate with the vector Φ a functional of functions with anticommuting values $L(\alpha^*, \alpha)$ (see (1)), defined by the formula

$$L = \sum_{m,n} \int L_{m,n}(k_1, \dots, k_m \mid l_1, \dots, l_n)\alpha^*(l_n) \dots \alpha^*(l_1)\alpha(k_m) \dots \alpha(k_1)\, d^m k\, d^n l,$$

where

$$L_{m,n}(k_1, \dots, k_m \mid l_1, \dots, l_n) =$$

$$= \frac{(-1)^m}{m!n!} (\hat{a}^+(k_1) \dots \hat{a}^+(k_m)\hat{a}(l_1) \dots \hat{a}(l_n)\Phi, \Phi);$$

however, for fermions one may also use the notation

$$L(\alpha^*, \alpha) = (e^{-\alpha\hat{a}^+} e^{\alpha^*\hat{a}}\Phi, \Phi)$$

with the corresponding clarification of its meaning. In what follows all formulas will be written in a form suitable both for bosons and for fermions.

We especially emphasize that the L-functional is defined for vectors from any space in which the canonical commutation relations are represented.

relations. Both in the Bose case and in the Fermi case one can prove that the L-functional exists for every vector from any representation of the commutation relations. (In other words, the L-functional admits arbitrary—not necessarily proper—canonical transformations.) However, in the Bose case the L-functional may fail to be analytic even if it corresponds to a vector from the Fock space H_a (a vector Φ corresponds to an L-functional differentiable infinitely many times provided that the sequence p_n of probabilities of finding n particles in the state Φ tends to zero faster than any power of $1/n$; in the case of a finite number of degrees of freedom this condition is also necessary for infinite differentiability of the L-functional).

If

$$\hat{b}(x) = \int \Phi(x, k)\, \hat{a}(k)\, dk + \int \Psi(x, k)\, \hat{a}^+(k)\, dk + f(x)$$

is a linear canonical transformation, and θ_a is the vacuum vector for the operators $\hat{a}(k)$ (i.e., $\hat{a}(k)\theta_a = 0$), then the L-functional on the vector θ_a, constructed by means of the operators $\hat{b}^+(x), \hat{b}(x)$, is equal to

$$\Lambda(\beta^*, \beta) = \exp\left(\frac{1}{2}\beta^*\Psi\Phi'\beta^* + \frac{1}{2}\beta\Phi^*\Psi^+\beta - \beta^*\Psi\Psi^+\beta - \beta f^* + \beta^* f\right)$$

(and exists independently of the propriety of the canonical transformation).

One can give simple formulas which make it possible to express the L-functional $L_1(\beta^*, \beta)$, constructed by means of the operators $\hat{b}^+(x), \hat{b}(x)$, in terms of the L-functional $L_0(a^*, a)$, constructed on the same vector by means of the operators $\hat{a}^+(k), \hat{a}(k)$; namely:

$$L_1(\beta^*, \beta) = \Lambda(\beta^*, \beta) L_0(\Phi'\beta^* - \Psi^+\beta, \Phi^+\beta - \Psi'\beta^*).$$

The L-functional can be defined not only for a pure state represented by a vector Φ, but also for a mixed state represented by a density matrix $\hat{K}$, acting in H, by the formula

$$L_{\hat{K}}(a^*, a) = \operatorname{Sp}\left(e^{-a^*\hat{a} + e a^* a \hat{a}} \hat{K}\right);$$

if the operator $\hat{K}$ acts in Fock space, then the formula holds

$$L_{\hat{K}}(a^*, a) = \int e^{\alpha a^* - \alpha^* a + \alpha a^* - a a^*} \tilde{K}(a^*, a)\, da\, da^*,$$

where $\tilde{K}(a^*, a)$ is the functional, defined in (1), corresponding to the matrix form of the operator $\hat{K}$, and also the formula

$$\tilde{K}(a^*, a) = \int e^{\alpha a^* + \alpha a^* - a a^* + a a^*} L_{\hat{K}}(\alpha^*, \alpha)\, d\alpha\, d\alpha^*;$$

continual integration in the Fermi case is understood in the sense of (1). From the formulas just indicated there follow in an obvious way formulas connecting the L-functional constructed on the vector $\Phi \in H_a$ with the Fock functional (1).

The conditions satisfied by the density matrix $\hat{K}$ impose certain restrictions on the corresponding functional L; in particular, from the condition $\operatorname{Sp} \hat{K} = 1$ it follows that $L(0, 0) = 1$, and from self-adjointness of the matrix $\hat{K}$ it follows that

$$L^*(a^*, a) = L(-a^*, -a)$$

(in the Fermi case this equality means that

$$L_{m,n}(k_1, \ldots, k_m \mid l_1, \ldots, l_n) = (-1)^{m+n} L_{n,m}(l_n, \ldots, l_1 \mid k_m, \ldots, k_1).$$

The condition of positive definiteness for L-functionals in the Bose case can be formulated as follows: for arbitrary numbers $\gamma_1, \ldots, \gamma_n$ and arbitrary functions $\alpha_1(k), \ldots, \alpha_n(k)$ the inequality must be satisfied

$$\sum_{1\leq i,j\leq n} \gamma_i^*\gamma_j e^{a_i a_j^*} L(a_i^* - a_j^*, a_i - a_j) \geq 0.$$

Functionals satisfying the three conditions listed can be interpreted as positive functionals on a certain algebra; such functionals have been considered by many authors (Segal, Araki and

etc.), in particular, in connection with the problem of describing representations of commutation relations.

Representing the states of a quantum-mechanical system by L-functionals, one can formulate quantum theory without using the concept of Hilbert space. In order to see this, let us first note that the mean value $\bar{A}$ of a physical quantity represented by the operator

$$\hat{A} = \sum_{m,n} \int A_{m,n}(k_1, \dots, k_m \mid l_1, \dots, l_n)\hat{a}^+(k_1) \dots \hat{a}^+(k_m)\hat{a}(l_1) \dots \hat{a}(l_n)\, d^m k\, d^n l,$$

in a state with density matrix $\hat{K}$, is equal to

$$\bar{A} = \sum_{m,n} (-1)^m m! n! \times$$

$$\times \int A_{m,n}(k_1, \dots, k_m \mid l_1, \dots, l_n) L_{m,n}(k_1, \dots, k_m \mid l_1, \dots, l_n)\, d^m k\, d^n l,$$

where $L_{m,n}(k_1, \dots, k_m \mid l_1, \dots, l_n)$ are the coefficient functions of the L-functional $L_{\hat{K}}(a^*, a)$ corresponding to the density matrix $\hat{K}$. The formula for the mean value of a physical quantity can be written in another form if the L-functional $L_{\hat{A}}(a^*, a)$, constructed with the aid of the operator $\hat{A}$, is known; namely, in the Bose case

$$\bar{A} = \int L_{\hat{A}}(-a^*, -a)\, L_{\hat{K}}(a^*, a)\, e^{-a^* a}\, da\, da^*.$$

Thus, knowing the L-functional, one can calculate all the physical characteristics of the state represented by this functional. Further, knowing the Hamiltonian of the system

$$\hat{H} = \sum_{m,n} \int H_{m,n}(k_1, \dots, k_m \mid l_1, \dots, l_n)\hat{a}^+(k_1) \dots, \hat{a}^+(k_m)\hat{a}(l_1) \dots$$

$$\dots \hat{a}(l_n)\, d^m k\, d^n l,$$

one can write the equation for the time variation of the L-functional in the form

$$i\frac{dL}{dt} = \sum_{m,n} \int H_{m,n}(k_1, \dots, k_m \mid l_1, \dots, l_n) \left[\left(a^*(k_1) - \frac{\delta}{\delta a(k_1)} \right) \dots \right.$$

$$\dots \left(a^*(k_m) - \frac{\delta}{\delta a(k_m)} \right) \frac{\delta}{\delta a^*(l_1)} \dots \frac{\delta}{\delta a^*(l_n)} - \left(\frac{\delta}{\delta a^*(l_1)} - a(l_1) \right) \dots$$

$$\left. \dots \left(\frac{\delta}{\delta a^*(l_n)} - a(l_n) \right) \left(-\frac{\delta}{\delta a(k_1)} \right) \dots \left(-\frac{\delta}{\delta a(k_m)} \right) \right] L \, d^m k \, d^n l$$

in the Bose case, and

$$i\frac{dL}{dt} = \sum_{m,n} \int H_{m,n}(k_1, \dots, k_m \mid l_1, \dots, l_n) \left[\left(a^*(k_1) - \frac{\delta}{\delta a(k_1)} \right) \dots \right.$$

$$\dots \left(a^*(k_m) - \frac{\delta}{\delta a(k_m)} \right) \frac{\delta}{\delta a^*(l_1)} \dots \frac{\delta}{\delta a^*(l_n)} L - L \left(-\frac{\delta}{\delta a(k_1)} \right) \dots$$

$$\left. \dots \left(-\frac{\delta}{\delta a(k_m)} \right) \left(\frac{\delta}{\delta a^*(l_1)} - a(l_1) \right) \dots \left(\frac{\delta}{\delta a^*(l_n)} - a(l_n) \right) \right] d^m k \, d^n l$$

in the Fermi case.

The formulation of quantum theory by means of L-functionals is, of course, equivalent to the usual one in the case where the latter is correct. However, in quantum field theory, as a rule, the Hamiltonian $\hat{H}$ is not a self-adjoint operator in Hilbert space and can be interpreted only as a self-adjoint formal polynomial in the creation and annihilation operators. This means that the operator $e^{-it\hat{H}}$, giving the solution of the equation of motion $i\, d\Phi/dt = \hat{H}\Phi$, does not exist. At the same time the equation of motion for the L-functional, written with the aid of the formal expression for $\hat{H}$, may turn out to be solvable, and the theory constructed—

in terms of L-functionals, is correct; this is always so if the equations for the Heisenberg operators are solvable.

As an example, let us consider a scalar field interacting with a source (2). In the presence of a form factor the Hamiltonian $\hat{H}$ is a self-adjoint operator in H_a, and the usual formulation of the theory is correct; however, in the case of a point source this is not so. The solution of the equation of motion for the L-functional has the form

$$L(\alpha^*, \alpha, t) =$$

$$= L(e^{-i\omega t}\alpha^*, e^{i\omega t}\alpha, 0) \exp\left[\alpha^*(e^{-i\omega t}-1)\lambda f/\omega^{-3/2} - \alpha(e^{i\omega t}-1)\lambda f/\omega^{3/2}\right]$$

and is quite correct also in the case of a point source, i.e., for $f(k) \equiv 1$. In subsequent publications more complicated models will be considered, in particular the BCS model.

Let us consider the question of locality of the theory in the formulation by means of L-functionals. In the Bose case we shall assume, in accordance with the situation usually encountered, that the Hamiltonian is expressed through Hermitian operators $\hat{\varphi}_i(x)$, $\hat{\pi}_i(x)$, satisfying the commutation relations

$$[\hat{\varphi}_i(x), \hat{\varphi}_j(x')] = [\hat{\pi}_i(x), \hat{\pi}_j(x')] = 0, \quad [\hat{\varphi}_i(x), \hat{\pi}^j(x')] = i\delta(x-x')\delta_i^j$$

(here x ranges over three-dimensional Euclidean space, and i takes a finite number of values). With the aid of the operators $\hat{\varphi}, \hat{\pi}$, for each state vector Φ we construct the functional

$$L(\alpha_i, \beta_i) = (\exp[i\sum \alpha_i\varphi_i] \exp[i\beta_i\pi_i]\Phi, \Phi),$$

where $\alpha_i(x), \beta_i(x)$ are real functions on three-dimensional Euclidean space. The functionals just introduced do not differ essentially from those defined above. If G is a domain in three-dimensional space, we define the G-part of the functional $L(\alpha_i, \beta_i)$ as the functional $L(\alpha_i, \beta_i)$ considered only on functions α_i, β_j with supports in the domain G. Let the Hamiltonian

$$\hat{H} = \sum_{m,n} \int H_{m,n}(x_1, \dots, x_m \mid y_1, \dots, y_n)\hat{\varphi}(x_1)\cdots\hat{\varphi}(x_m)\hat{\pi}(y_1)\cdots$$

$$\cdots\hat{\pi}(y_n)\, d^{3m}x\, d^{3n}y$$

be a local operator (i.e., the functions $H_{m,n}$ are concentrated on sets of points of the form $(x, \dots, x \mid x, \dots, x)$). Then dL_G/dt is expressed through L_G (if the Hamiltonian is only ε-local, i.e., $H_{m,n} = 0$ in the case when at least one pair of arguments is separated by a distance $> \varepsilon$, then dL_G/dt is expressed through $L_{O_\varepsilon(G)}$, where $O_\varepsilon(G)$ is the ε-neighborhood of the domain G). The proof is based on the observation that, in the expression for dL/dt, the terms containing only variational derivatives mutually cancel. In the Fermi case one can prove analogous results; here one should use the former definition of the L-functional.

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CITED LITERATURE

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2. S. Schweber, *Introduction to Relativistic Quantum Field Theory*, IL, 1963.

Note: Figure translations are in progress. See original paper for figures.