The topological equivalence of systems of total differential equations in neighborhoods of closed trajectories

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Abstract

A system of equations

$$dx = \sum_{i=1}^{n} p_i(x)\, dt^i, \tag{1}$$

is considered, where x and $p_i(x)$ are $(n+1)$-dimensional vectors for which the conditions of complete integrability are satisfied. It is assumed that the system (1) possesses a closed trajectory γ. It is proved that at least one of the vectors $p_i(x)$ is non-zero at all points of this closed trajectory.

In the neighborhood of γ, a system of local coordinates (z, s) is introduced, where z is an n-dimensional vector and s is a scalar. It is shown that the conditions of complete integrability are also satisfied for the system corresponding to system (1) in these local coordinates. The simplest case is examined, where the system (1) in local coordinates corresponds to a linear system

$$dz = \sum_{i=1}^{n} B_i(s) z\, dt^i + A(s) z\, ds, \tag{2}$$

where $B_i(s)$ and $A(s)$ are $n \times n$ matrices with period 1. For the case where the Jordan normal form for the matrices $B_i(0)$ $(i = 1, 2, \dots, n-1)$ is diagonal, conditions are provided under which two systems of the form (1) possessing closed trajectories are topologically equivalent in the neighborhoods of these trajectories.

Bibliography: 8 items.

Full Text

Preamble

This work investigates the properties of systems of differential equations of the form $\frac{dx}{dt_i} = p_i(x)$ for $i = 1, \dots, n$, following the foundational approaches established in [**?**, **?**, **?**]. We consider the $(n+1)$-dimensional system:

$$\frac{dx}{dt_i} = p_i(x), \quad i = 1, 2, \dots, n$$

where the functions $p_i(x)$ satisfy the commutativity condition:

$$\frac{\partial p_i(x)}{\partial x} p_j(x) = \frac{\partial p_j(x)}{\partial x} p_i(x)$$

for all $i, j = 1, \dots, n$. Here, x belongs to a domain D in $(n+1)$-dimensional space, and $p_i(x)$ are assumed to be of class C^r $(r \geq 1)$.

Section 1. Structural Properties and Commutativity

Let Q be a subdomain where $p_i(x) \neq 0$. For any $x_0 \in Q$, there exist functions $k_j(x)$ such that $p_j(x) = k_j(x) p_i(x)$. From the commutativity conditions, it follows that:

$$\frac{dk_j(x)}{dx} p_i(x) = 0$$

This implies that the coefficients $k_j(x)$ are constant along the trajectories of the system. Consequently, in the domain Q, the vectors $p_j(x)$ are proportional to $p_i(x)$, which simplifies the integration of the system. If we consider a sequence of subdomains Q_s $(s = 1, 2, \dots, m)$, the relationship between the vector fields can be expressed as $p_{is}(x) = M(s) p_{is}(x)$, where $M(s)$ is a transition matrix.

Section 2. Transformation and Integration

We consider the transformation of the system using the variables z and s. The derivatives with respect to these variables are governed by the following relations:

$$\frac{\partial q_i(z,s)}{\partial z} q_j(z,s) = \theta^*(s) \left(\frac{\partial p_i(x)}{\partial x} - \frac{\Pi_i(x)}{\Pi_n(x)} \frac{\partial p_n(x)}{\partial x} - p_n(x) \frac{d}{dx} \left(\frac{\Pi_i(x)}{\Pi_n(x)} \right) \right) \times \left(p_j(x) - \frac{\Pi_j(x)}{\Pi_n(x)} p_n(x) \right)$$

By applying the conditions from (50) and (51), we establish the equivalence of the mixed partial derivatives. The system can then be reduced to the form:

$$dz = \sum_{i=1}^{n-1} B_i(s) z \, dt_i + A(s) z \, ds$$

where $B_i(s)$ are $n \times n$ matrices. The consistency of this system requires that the matrices satisfy the commutation relation $C B_i(0) = B_i(0) C$, where $C = Z(1)$ is the fundamental matrix solution at $s = 1$.

Section 3. Global Solutions and Mapping

The general solution of the transformed system (52) can be expressed using the exponential mapping. Specifically, if $Z(s)$ is the solution to the initial value problem $dz/ds = A(s) z$ with $Z(0) = E$, then the state at any point can be related to the initial state z_0 via:

$$z(s,t) = Z(s) \exp \left(\sum B_i(0) t_i \right) z_0$$

This formulation allows us to map the solutions across different regions of the domain. We define a mapping $\mu_x : z \to z'$ such that the components transform as $z'_j = \operatorname{sign}(z_j) |z_j|^{\chi}$.

Section 4. Convergence and Stability Conditions

We analyze the behavior of the system as $x \to \infty$ or $s \to 0$. The existence of a stable solution depends on the signs of the coefficients α_j and the magnitudes of the eigenvalues of the matrices B_i. Specifically, if $\prod |c_j|^{\alpha_j} = 1$, the system exhibits specific asymptotic properties.

For the case where $k = n - 1$, the conditions for the existence of a unique solution simplify significantly. The mapping $v(z, s)$ defined by:

$$v(z, s) = Z(s)\mu_x Z^{-1}(s)z$$

provides a continuous link between the states at $s = 0$ and $s = 1$. This ensures that the structural properties of the differential equations are preserved under the group of transformations defined by (52) and (57).

References

1. Krasovskii I.P., *On the Theory of Stability of Motion*, 1958, pp. 37–42.
2. Lur’ e L.S., *Analytical Mechanics*, 1964, 63 (105):3, pp. 392–408.
3. *Journal of Differential Equations*, No. 8, pp. 1282–1291, 1967.
4. *Mathematical Modeling*, Vol. 3, Moscow, 1964.
5. *Numerical Methods*, Moscow-Leningrad, 1948.
6. *Russian Academy of Sciences Reports*, No. 1, pp. 59–65, 1964.
7. De la Vallée Poussin C.J., *Cours d’Analyse Infinitésimale*, Vol. II, Louvain, 1925.
8. *Moscow State University Bulletin*, 23 (65):2, pp. 161–186, 1948.

Figures

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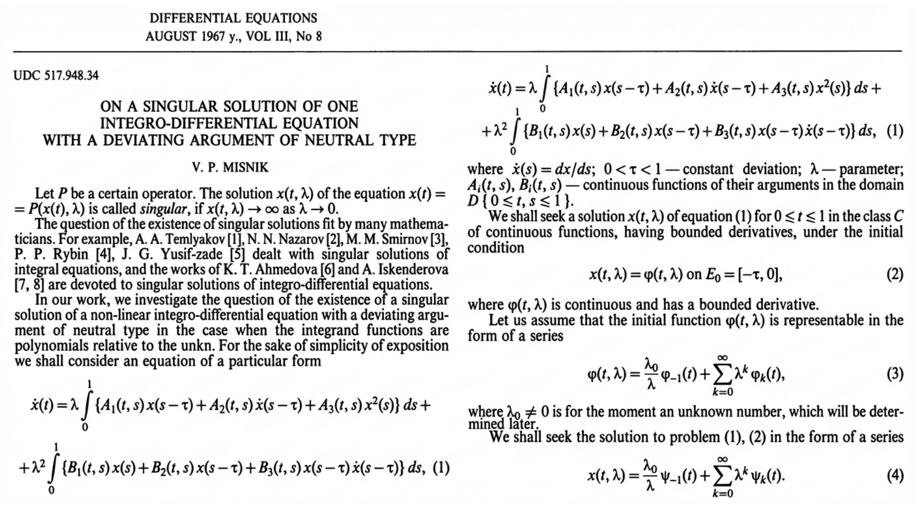

DIFFERENTIAL EQUATIONS
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ON A SINGULAR SOLUTION OF ONE INTEGRO-DIFFERENTIAL EQUATION WITH A DEVIATING ARGUMENT OF NEUTRAL TYPE

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Let P be a certain operator. The solution $x(t, \lambda)$ of the equation $x(t) = P(x(t), \lambda)$ is called *singular*, if $x(t, \lambda) \to \infty$ as $\lambda \to 0$.

The question of the existence of singular solutions fit by many mathematicians. For example, A. A. Temlyakov [1], N. N. Nazarov [2], M. M. Smirnov [3], P. P. Rybin [4], J. G. Yusif-zade [5] dealt with singular solutions of integral equations, and the works of K. T. Ahmedova [6] and A. Iskenderova [7, 8] are devoted to singular solutions of integro-differential equations.

In our work, we investigate the question of the existence of a singular solution of a non-linear integro-differential equation with a deviating argument of neutral type in the case when the integrand functions are polynomials relative to the unkn. For the sake of simplicity of exposition we shall consider an equation of a particular form

$$\dot{x}(t) = \lambda \int_0^1 \{A_1(t, s)\, x(s-\tau) + A_2(t, s)\, \dot{x}(s-\tau) + A_3(t, s)\, x^2(s)\}\, ds +$$

$$+ \lambda^2 \int_0^1 \{B_1(t, s)\, x(s) + B_2(t, s)\, x(s-\tau) + B_3(t, s)\, x(s-\tau)\, \dot{x}(s-\tau)\}\, ds, \quad (1)$$

$$\dot{x}(t) = \lambda \int_0^1 \{A_1(t, s)\, x(s-\tau) + A_2(t, s)\, \dot{x}(s-\tau) + A_3(t, s)\, x^2(s)\}\, ds +$$

$$+ \lambda^2 \int_0^1 \{B_1(t, s)\, x(s) + B_2(t, s)\, x(s-\tau) + B_3(t, s)\, x(s-\tau)\, \dot{x}(s-\tau)\}\, ds, \quad (1)$$

where $\dot{x}(s) = dx/ds$; $0 < \tau < 1$ — constant deviation; λ — parameter; $A_i(t, s)$, $B_i(t, s)$ — continuous functions of their arguments in the domain $D\,\{0 \leq t, s \leq 1\}$.

We shall seek a solution $x(t, \lambda)$ of equation (1) for $0 \leq t \leq 1$ in the class C of continuous functions, having bounded derivatives, under the initial condition

$$x(t, \lambda) = \varphi(t, \lambda) \text{ on } E_0 = [-\tau, 0], \quad (2)$$

where $\varphi(t, \lambda)$ is continuous and has a bounded derivative.

Let us assume that the initial function $\varphi(t, \lambda)$ is representable in the form of a series

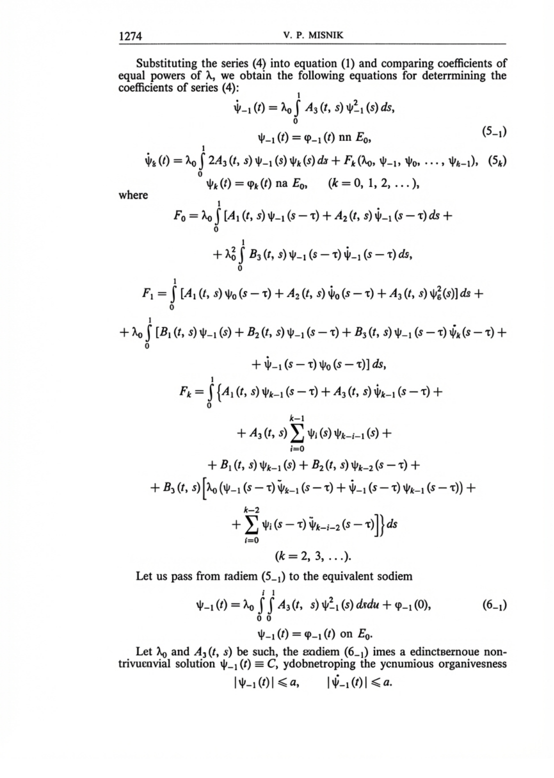
$$\varphi(t, \lambda) = \frac{\lambda_0}{\lambda}\, \varphi_{-1}(t) + \sum_{k=0}^{\infty} \lambda^k\, \varphi_k(t), \quad (3)$$

where $\lambda_0 \neq 0$ is for the moment an unknown number, which will be determined later.

We shall seek the solution to problem (1), (2) in the form of a series

$$x(t, \lambda) = \frac{\lambda_0}{\lambda}\, \psi_{-1}(t) + \sum_{k=0}^{\infty} \lambda^k\, \psi_k(t). \quad (4)$$

Figure 1: Figure 1



Substituting the series (4) into equation (1) and comparing coefficients of equal powers of λ, we obtain the following equations for deterrmining the coefficients of series (4):

$$\dot{\psi}_{-1}(t) = \lambda_0 \int_0^1 A_3(t, s)\, \psi_{-1}^2(s)\, ds,$$

$$\psi_{-1}(t) = \varphi_{-1}(t) \text{ nn } E_0, \quad (5_{-1})$$

$$\dot{\psi}_k(t) = \lambda_0 \int_0^1 2A_3(t, s)\, \psi_{-1}(s)\, \psi_k(s)\, ds + F_k(\lambda_0, \psi_{-1}, \psi_0, \ldots, \psi_{k-1}), \quad (5_k)$$

$$\psi_k(t) = \varphi_k(t) \text{ na } E_0, \qquad (k = 0, 1, 2, \ldots),$$

where

$$F_0 = \lambda_0 \int_0^1 [A_1(t, s)\, \psi_{-1}(s-\tau) + A_2(t, s)\, \dot{\psi}_{-1}(s-\tau)\, ds +$$

$$+ \lambda_0^2 \int_0^1 B_3(t, s)\, \psi_{-1}(s-\tau)\, \dot{\psi}_{-1}(s-\tau)\, ds,$$

$$F_1 = \int_0^1 [A_1(t, s)\, \psi_0(s-\tau) + A_2(t, s)\, \dot{\psi}_0(s-\tau) + A_3(t, s)\, \psi_0^2(s)]\, ds +$$

$$+ \lambda_0 \int_0^1 [B_1(t, s)\, \psi_{-1}(s) + B_2(t, s)\, \psi_{-1}(s-\tau) + B_3(t, s)\, \psi_{-1}(s-\tau)\, \dot{\psi}_k(s-\tau) +$$

$$+ \dot{\psi}_{-1}(s-\tau)\, \psi_0(s-\tau)]\, ds,$$

$$F_k = \int_0^1 \Big\{A_1(t, s)\, \psi_{k-1}(s-\tau) + A_3(t, s)\, \dot{\psi}_{k-1}(s-\tau) +$$

$$+ A_3(t, s) \sum_{i=0}^{k-1} \psi_i(s)\, \psi_{k-i-1}(s) +$$

$$+ B_1(t, s)\, \psi_{k-1}(s) + B_2(t, s)\, \psi_{k-2}(s-\tau) +$$

$$+ B_3(t, s)\Big[\lambda_0 (\psi_{-1}(s-\tau)\, \check{\psi}_{k-1}(s-\tau) + \dot{\psi}_{-1}(s-\tau)\, \psi_{k-1}(s-\tau)) +$$

$$+ \sum_{i=0}^{k-2} \psi_i(s-\tau)\, \check{\psi}_{k-i-2}(s-\tau)\Big]\Big\}\, ds$$

$$(k = 2, 3, \ldots).$$

Let us pass from radiem (5_{-1}) to the equivalent sodiem

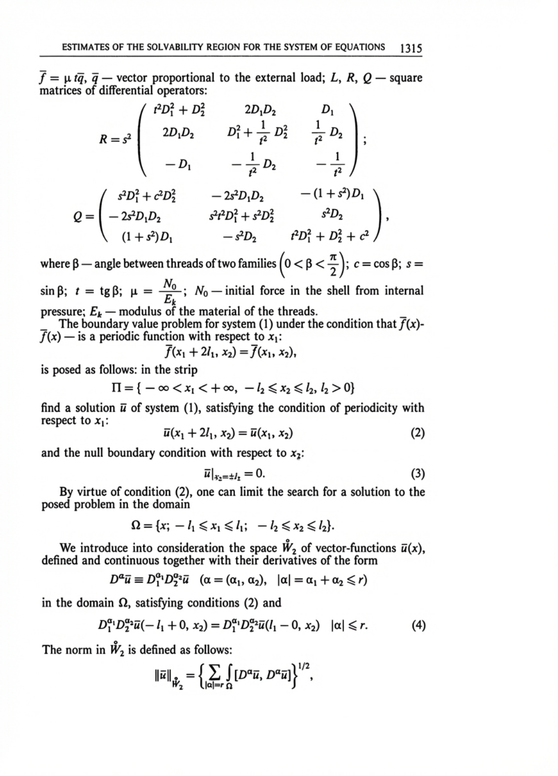
$$\psi_{-1}(t) = \lambda_0 \int_0^t \int_0^1 A_3(t, s)\, \psi_{-1}^2(s)\, ds\,du + \varphi_{-1}(0), \quad (6_{-1})$$

$$\psi_{-1}(t) = \varphi_{-1}(t) \text{ on } E_0.$$

Let λ_0 and $A_3(t, s)$ be such, the ead iem (6_{-1}) imes a edinctвernoue nontrivuenvial solution $\psi_{-1}(t) \equiv C$, ydobnetroping the ycnumious organivesness

$$|\psi_{-1}(t)| \leqslant a, \qquad |\dot{\psi}_{-1}(t)| \leqslant a.$$

Figure 2: Figure 2



$\overline{f} = \mu\, t\overline{q}$, $\overline{q}$ — vector proportional to the external load; L, R, Q — square matrices of differential operators:

$$R = s^2 \begin{pmatrix} t^2D_1^2 + D_2^2 & 2D_1D_2 & D_1 \\ 2D_1D_2 & D_1^2 + \frac{1}{t^2} D_2^2 & \frac{1}{t^2} D_2 \\ -D_1 & -\frac{1}{t^2} D_2 & -\frac{1}{t^2} \end{pmatrix};$$

$$Q = \begin{pmatrix} s^2D_1^2 + c^2D_2^2 & -2s^2D_1D_2 & -(1+s^2)D_1 \\ -2s^2D_1D_2 & s^2t^2D_1^2 + s^2D_2^2 & s^2D_2 \\ (1+s^2)D_1 & -s^2D_2 & t^2D_1^2 + D_2^2 + c^2 \end{pmatrix},$$

where β — angle between threads of two families $\left(0 < \beta < \frac{\pi}{2}\right)$; $c = \cos\beta$; $s = \sin\beta$; $t = \operatorname{tg}\beta$; $\mu = \frac{N_0}{E_k}$; N_0 — initial force in the shell from internal pressure; E_k — modulus of the material of the threads.

The boundary value problem for system (1) under the condition that $\overline{f}(x)$-$\overline{f}(x)$ — is a periodic function with respect to x_1:

$$\overline{f}(x_1 + 2l_1, x_2) = \overline{f}(x_1, x_2),$$

is posed as follows: in the strip

$$\Pi = \{-\infty < x_1 < +\infty,\ -l_2 \leqslant x_2 \leqslant l_2,\ l_2 > 0\}$$

find a solution $\overline{u}$ of system (1), satisfying the condition of periodicity with respect to x_1:

$$\overline{u}(x_1 + 2l_1, x_2) = \overline{u}(x_1, x_2) \tag{2}$$

and the null boundary condition with respect to x_2:

$$\overline{u}|_{x_2 = \pm l_2} = 0. \tag{3}$$

By virtue of condition (2), one can limit the search for a solution to the posed problem in the domain

$$\Omega = \{x;\ -l_1 \leqslant x_1 \leqslant l_1;\ \ -l_2 \leqslant x_2 \leqslant l_2\}.$$

We introduce into consideration the space $\mathring{W}_2$ of vector-functions $\overline{u}(x)$, defined and continuous together with their derivatives of the form

$$D^\alpha \overline{u} \equiv D_1^{\alpha_1} D_2^{\alpha_2} \overline{u} \quad (\alpha = (\alpha_1, \alpha_2),\ \ |\alpha| = \alpha_1 + \alpha_2 \leqslant r)$$

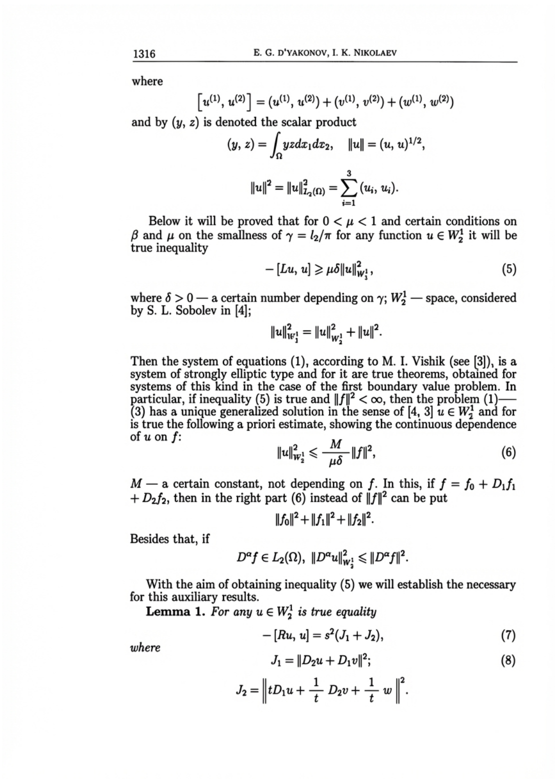
in the domain Ω, satisfying conditions (2) and

$$D_1^{\alpha_1} D_2^{\alpha_2} \overline{u}(-l_1 + 0, x_2) = D_1^{\alpha_1} D_2^{\alpha_2} \overline{u}(l_1 - 0, x_2) \quad |\alpha| \leqslant r. \tag{4}$$

The norm in $\mathring{W}_2$ is defined as follows:

$$\|\overline{u}\|_{\mathring{W}_2} = \left\{ \sum_{|\alpha| = r} \int_\Omega [D^\alpha \overline{u}, D^\alpha \overline{u}] \right\}^{1/2},$$

Figure 3: Figure 3



where

$$\left[u^{(1)}, u^{(2)}\right] = (u^{(1)}, u^{(2)}) + (v^{(1)}, v^{(2)}) + (w^{(1)}, w^{(2)})$$

and by (y, z) is denoted the scalar product

$$(y, z) = \int_\Omega yz\,dx_1dx_2, \quad \|u\| = (u, u)^{1/2},$$

$$\|u\|^2 = \|u\|^2_{L_2(\Omega)} = \sum_{i=1}^{3}(u_i, u_i).$$

Below it will be proved that for $0 < \mu < 1$ and certain conditions on β and μ on the smallness of $\gamma = l_2/\pi$ for any function $u \in W_2^1$ it will be true inequality

$$-[Lu, u] \geqslant \mu\delta\|u\|^2_{W_1^1}, \tag{5}$$

where $\delta > 0$ — a certain number depending on γ; W_2^1 — space, considered by S. L. Sobolev in [4];

$$\|u\|^2_{W_1^1} = \|u\|^2_{W_2^1} + \|u\|^2.$$

Then the system of equations (1), according to M. I. Vishik (see [3]), is a system of strongly elliptic type and for it are true theorems, obtained for systems of this kind in the case of the first boundary value problem. In particular, if inequality (5) is true and $\|f\|^2 < \infty$, then the problem (1)—(3) has a unique generalized solution in the sense of [4, 3] $u \in W_2^1$ and for is true the following a priori estimate, showing the continuous dependence of u on f:

$$\|u\|^2_{W_2^1} \leqslant \frac{M}{\mu\delta}\|f\|^2, \tag{6}$$

M — a certain constant, not depending on f. In this, if $f = f_0 + D_1f_1 + D_2f_2$, then in the right part (6) instead of $\|f\|^2$ can be put

$$\|f_0\|^2 + \|f_1\|^2 + \|f_2\|^2.$$

Besides that, if

$$D^\alpha f \in L_2(\Omega), \ \|D^\alpha u\|^2_{W_2^1} \leqslant \|D^\alpha f\|^2.$$

With the aim of obtaining inequality (5) we will establish the necessary for this auxiliary results.

Lemma 1. *For any $u \in W_2^1$ is true equality*

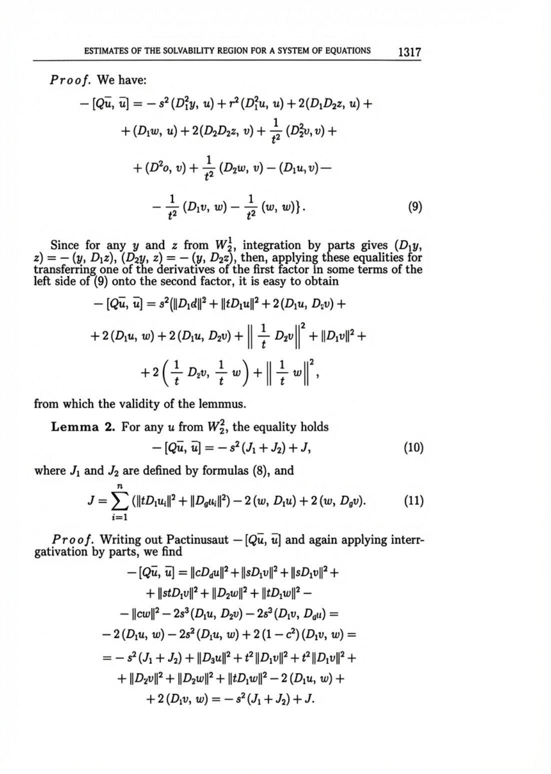
$$-[Ru, u] = s^2(J_1 + J_2), \tag{7}$$

where

$$J_1 = \|D_2u + D_1v\|^2; \tag{8}$$

$$J_2 = \left\|tD_1u + \frac{1}{t}D_2v + \frac{1}{t}w\right\|^2.$$

Figure 4: Figure 4



Proof. We have:

$$-[Q\bar{u},\ \bar{u}] = -s^2(D_1^2 y,\ u) + r^2(D_1^2 u,\ u) + 2(D_1 D_2 z,\ u) +$$
$$+ (D_1 w,\ u) + 2(D_2 D_2 z,\ v) + \frac{1}{t^2}(D_2^2 v, v) +$$
$$+ (D^2 o,\ v) + \frac{1}{t^2}(D_2 w,\ v) - (D_1 u, v) -$$
$$-\frac{1}{t^2}(D_1 v,\ w) - \frac{1}{t^2}(w,\ w)\}. \tag{9}$$

Since for any y and z from W_2^1, integration by parts gives $(D_1 y,\ z) = -(y,\ D_1 z)$, $(D_2 y,\ z) = -(y,\ D_2 z)$, then, applying these equalities for transferring one of the derivatives of the first factor in some terms of the left side of (9) onto the second factor, it is easy to obtain

$$-[Q\bar{u},\ \bar{u}] = s^2(\|D_1 d\|^2 + \|tD_1 u\|^2 + 2(D_1 u,\ D_2 v) +$$
$$+ 2(D_1 u,\ w) + 2(D_1 u,\ D_2 v) + \left\| \frac{1}{t} D_2 v \right\|^2 + \|D_1 v\|^2 +$$
$$+ 2\left(\frac{1}{t} D_2 v,\ \frac{1}{t} w\right) + \left\| \frac{1}{t} w \right\|^2,$$

from which the validity of the lemmus.

Lemma 2. For any u from W_2^2, the equality holds

$$-[Q\bar{u},\ \bar{u}] = -s^2(J_1 + J_2) + J, \tag{10}$$

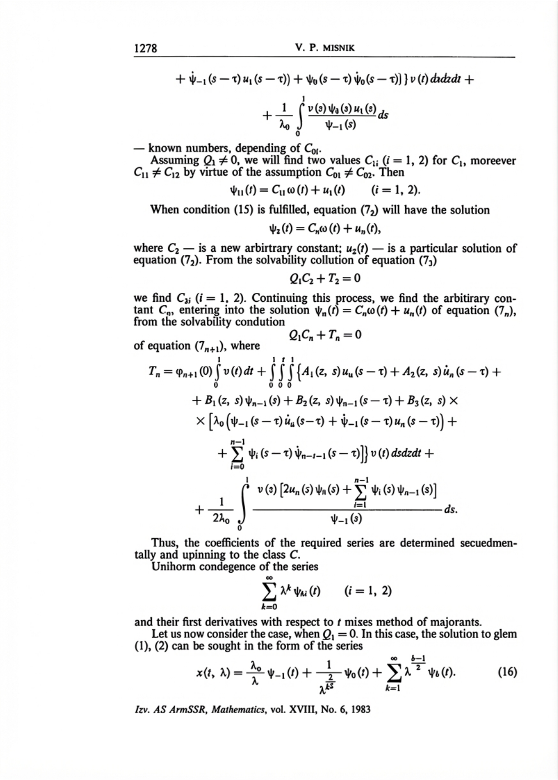
where J_1 and J_2 are defined by formulas (8), and

$$J = \sum_{i=1}^{n} (\|tD_1 u_i\|^2 + \|D_g u_i\|^2) - 2(w,\ D_1 u) + 2(w,\ D_g v). \tag{11}$$

Proof. Writing out Pactinusaut $-[Q\bar{u},\ \bar{u}]$ and again applying interrgativation by parts, we find

$$-[Q\bar{u},\ \bar{u}] = \|cD_d u\|^2 + \|sD_1 v\|^2 + \|sD_1 v\|^2 +$$
$$+ \|stD_1 v\|^2 + \|D_2 w\|^2 + \|tD_1 w\|^2 -$$
$$- \|cw\|^2 - 2s^3(D_1 u,\ D_2 v) - 2s^3(D_1 v,\ D_d u) =$$
$$-2(D_1 u,\ w) - 2s^2(D_1 u,\ w) + 2(1 - c^2)(D_1 v,\ w) =$$
$$= -s^2(J_1 + J_2) + \|D_3 u\|^2 + t^2\|D_1 v\|^2 + t^2\|D_1 v\|^2 +$$
$$+ \|D_2 v\|^2 + \|D_2 w\|^2 + \|tD_1 w\|^2 - 2(D_1 u,\ w) +$$
$$+ 2(D_1 v,\ w) = -s^2(J_1 + J_2) + J.$$

Figure 5: Figure 5



$$+\dot{\psi}_{-1}(s-\tau)\,u_1(s-\tau)) + \psi_0(s-\tau)\,\dot{\psi}_0(s-\tau)]\}\,v(t)\,dsdzdt +$$

$$+\frac{1}{\lambda_0}\int_0^1 \frac{v(s)\,\psi_0(s)\,u_1(s)}{\psi_{-1}(s)}\,ds$$

— known numbers, depending of C_{0i}.

Assuming $Q_1 \neq 0$, we will find two values C_{1i} $(i = 1, 2)$ for C_1, moreever $C_{11} \neq C_{12}$ by virtue of the assumption $C_{01} \neq C_{02}$. Then

$$\psi_{11}(t) = C_{11}\,\omega(t) + u_1(t) \qquad (i = 1,\ 2).$$

When condition (15) is fulfilled, equation (7_2) will have the solution

$$\psi_2(t) = C_n\omega(t) + u_n(t),$$

where C_2 — is a new arbirtrary constant; $u_2(t)$ — is a particular solution of equation (7_2). From the solvability collution of equation (7_3)

$$Q_1C_2 + T_2 = 0$$

we find C_{3i} $(i = 1,\ 2)$. Continuing this process, we find the arbitirary contant C_n, entering into the solution $\psi_n(t) = C_n\omega(t) + u_n(t)$ of equation (7_n), from the solvability condution

$$Q_1C_n + T_n = 0$$

of equation (7_{n+1}), where

$$T_n = \varphi_{n+1}(0)\int_0^1 v(t)\,dt + \int_0^1\int_0^t\int_0^1 \{A_1(z,\ s)\,u_u(s-\tau) + A_2(z,\ s)\,\dot{u}_n(s-\tau) +$$

$$+ B_1(z,\ s)\,\psi_{n-1}(s) + B_2(z,\ s)\,\psi_{n-1}(s-\tau) + B_3(z,\ s)\,\times$$

$$\times\left[\lambda_0\left(\psi_{-1}(s-\tau)\,\dot{u}_u(s-\tau) + \dot{\psi}_{-1}(s-\tau)\,u_n(s-\tau)\right) +\right.$$

$$\left.+\sum_{i=0}^{n-1}\psi_i(s-\tau)\,\dot{\psi}_{n-i-1}(s-\tau)\right]\Big\}\,v(t)\,dsdzdt +$$

$$+\frac{1}{2\lambda_0}\int_0^1 \frac{v(s)\left[2u_n(s)\,\psi_n(s) + \sum\limits_{i=1}^{n-1}\psi_i(s)\,\psi_{n-1}(s)\right]}{\psi_{-1}(s)}\,ds.$$

Thus, the coefficients of the required series are determined secuedmentally and upinning to the class C.

Unihorm condegence of the series

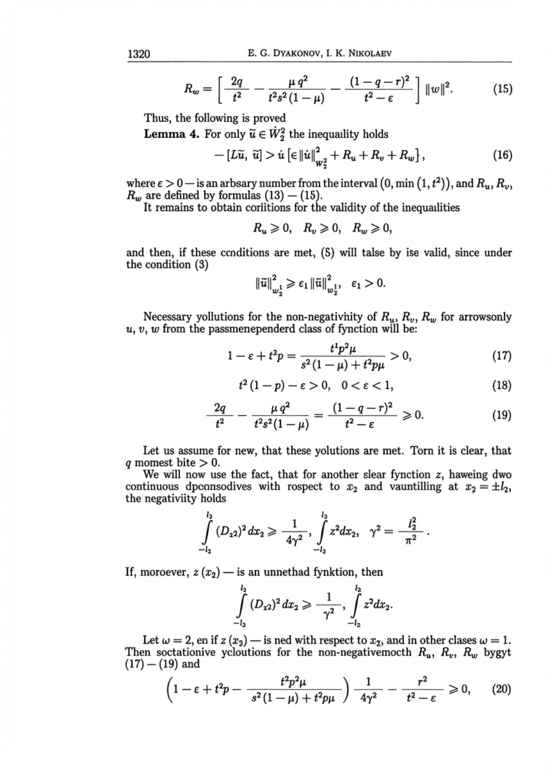
$$\sum_{k=0}^{\infty}\lambda^k\,\psi_{ki}(t) \qquad (i = 1,\ 2)$$

and their first derivatives with respect to t mises method of majorants.

Let us now consider the case, when $Q_1 = 0$. In this case, the solution to glem (1), (2) can be sought in the form of the series

$$x(t,\ \lambda) = \frac{\lambda_0}{\lambda}\,\psi_{-1}(t) + \frac{1}{\lambda^{\frac{2}{kS}}}\,\psi_0(t) + \sum_{k=1}^{\infty}\lambda^{\frac{b-1}{2}}\,\psi_b(t). \qquad (16)$$

Figure 6: Figure 6



$$R_w = \left[\frac{2q}{t^2} - \frac{\mu q^2}{t^2 s^2 (1-\mu)} - \frac{(1-q-r)^2}{t^2-\varepsilon} \right] \|w\|^2. \tag{15}$$

Thus, the following is proved

Lemma 4. For only $\tilde{u} \in \dot{W}_2^2$ the inequaility holds

$$-[L\tilde{u}, \tilde{u}] > \dot{u}\left[\epsilon \|\dot{u}\|^2_{W_2^2} + R_u + R_v + R_w\right\}, \tag{16}$$

where $\varepsilon > 0$ — is an arbsary number from the interval $(0, \min(1, t^2))$, and R_u, R_v, R_w are defined by formulas (13) — (15).

It remains to obtain corlitions for the validity of the inequailities

$$R_u \geqslant 0, \quad R_v \geqslant 0, \quad R_w \geqslant 0,$$

and then, if these ccnditions are met, (5) will talse by ise valid, since under the condition (3)

$$\|\bar{u}\|^2_{w_2^1} \geqslant \varepsilon_1 \|\bar{u}\|^2_{w_2^1}, \quad \varepsilon_1 > 0.$$

Necessary yollutions for the non-negativnity of R_u, R_v, R_w for arrowsonly u, v, w from the passmenependerd class of fynction will be:

$$1 - \varepsilon + t^2 p = \frac{t^1 p^2 \mu}{s^2(1-\mu) + t^2 p\mu} > 0, \tag{17}$$

$$t^2(1-p) - \varepsilon > 0, \quad 0 < \varepsilon < 1, \tag{18}$$

$$\frac{2q}{t^2} - \frac{\mu q^2}{t^2 s^2 (1-\mu)} = \frac{(1-q-r)^2}{t^2-\varepsilon} \geqslant 0. \tag{19}$$

Let us assume for new, that these yolutions are met. Torn it is clear, that q momest bite > 0.

We will now use the fact, that for another slear fynction z, haweing dwo continuous dpconsodives with rospect to x_2 and vauntilling at $x_2 = \pm l_2$, the negativiity holds

$$\int_{-l_2}^{l_3} (D_{z2})^2 \, dx_2 \geqslant \frac{1}{4\gamma^2}, \int_{-l_2}^{l_3} z^2 dx_2, \quad \gamma^2 = \frac{l_2^2}{\pi^2}.$$

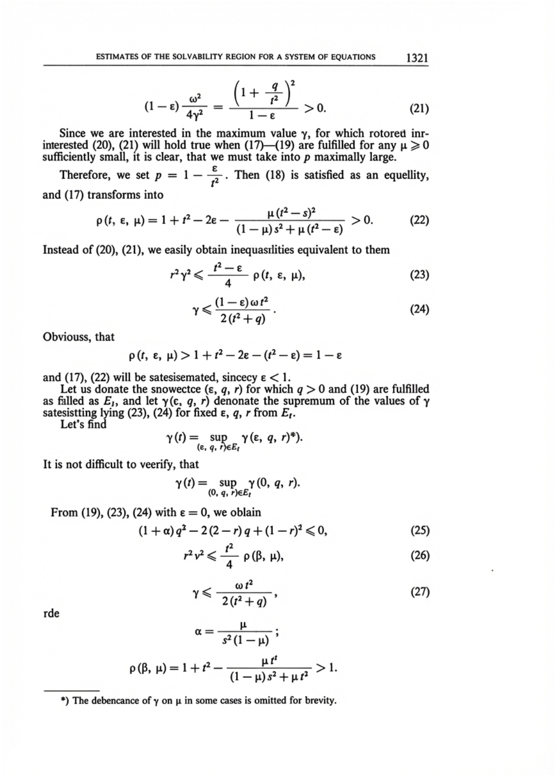
If, moroever, $z(x_2)$ — is an unnethad fynktion, then

$$\int_{-l_2}^{l_2} (D_{x2})^2 \, dx_2 \geqslant \frac{1}{\gamma^2}, \int_{-l_2}^{l_2} z^2 dx_2.$$

Let $\omega = 2$, en if $z(x_3)$ — is ned with respect to x_3, and in other clases $\omega = 1$. Then soctationive ycloutions for the non-negativemocth R_u, R_v, R_w bygyt (17) — (19) and

$$\left(1 - \varepsilon + t^2 p - \frac{t^2 p^2 \mu}{s^2(1-\mu) + t^2 p\mu}\right) \frac{1}{4\gamma^2} - \frac{r^2}{t^2-\varepsilon} \geqslant 0, \tag{20}$$

Figure 7: Figure 7



$$(1-\varepsilon)\frac{\omega^2}{4\gamma^2} = \frac{\left(1+\frac{q}{t^2}\right)^2}{1-\varepsilon} > 0. \tag{21}$$

Since we are interested in the maximum value γ, for which rotored inr-interested (20), (21) will hold true when (17)—(19) are fulfilled for any $\mu \geqslant 0$ sufficiently small, it is clear, that we must take into p maximally large.

Therefore, we set $p = 1 - \frac{\varepsilon}{t^2}$. Then (18) is satisfied as an equellity, and (17) transforms into

$$\rho(t, \varepsilon, \mu) = 1 + t^2 - 2\varepsilon - \frac{\mu(t^2-s)^2}{(1-\mu)s^2 + \mu(t^2-\varepsilon)} > 0. \tag{22}$$

Instead of (20), (21), we easily obtain inequasrlities equivalent to them

$$r^2\gamma^2 \leqslant \frac{t^2-\varepsilon}{4}\rho(t, \varepsilon, \mu), \tag{23}$$

$$\gamma \leqslant \frac{(1-\varepsilon)\omega t^2}{2(t^2+q)}. \tag{24}$$

Obviouss, that

$$\rho(t, \varepsilon, \mu) > 1 + t^2 - 2\varepsilon - (t^2 - \varepsilon) = 1 - \varepsilon$$

and (17), (22) will be satesisemated, sincecy $\varepsilon < 1$.

Let us donate the snowectce (ε, q, r) for which $q > 0$ and (19) are fulfilled as fialled as E_t, and let $\gamma(\varepsilon, q, r)$ denonate the supremum of the values of γ satesistting lying (23), (24) for fixed ε, q, r from E_t.

Let's find

$$\gamma(t) = \sup_{(\varepsilon, q, r)\in E_t} \gamma(\varepsilon, q, r)^{*)}.$$

It is not difficult to veerify, that

$$\gamma(t) = \sup_{(0, q, r)\in E_t} \gamma(0, q, r).$$

From (19), (23), (24) with $\varepsilon = 0$, we oblain

$$(1+\alpha)q^2 - 2(2-r)q + (1-r)^2 \leqslant 0, \tag{25}$$

$$r^2 v^2 \leqslant \frac{t^2}{4}\rho(\beta, \mu), \tag{26}$$

$$\gamma \leqslant \frac{\omega t^2}{2(t^2+q)}, \tag{27}$$

rde

$$\alpha = \frac{\mu}{s^2(1-\mu)};$$

$$\rho(\beta, \mu) = 1 + t^2 - \frac{\mu t^t}{(1-\mu)s^2 + \mu t^2} > 1.$$

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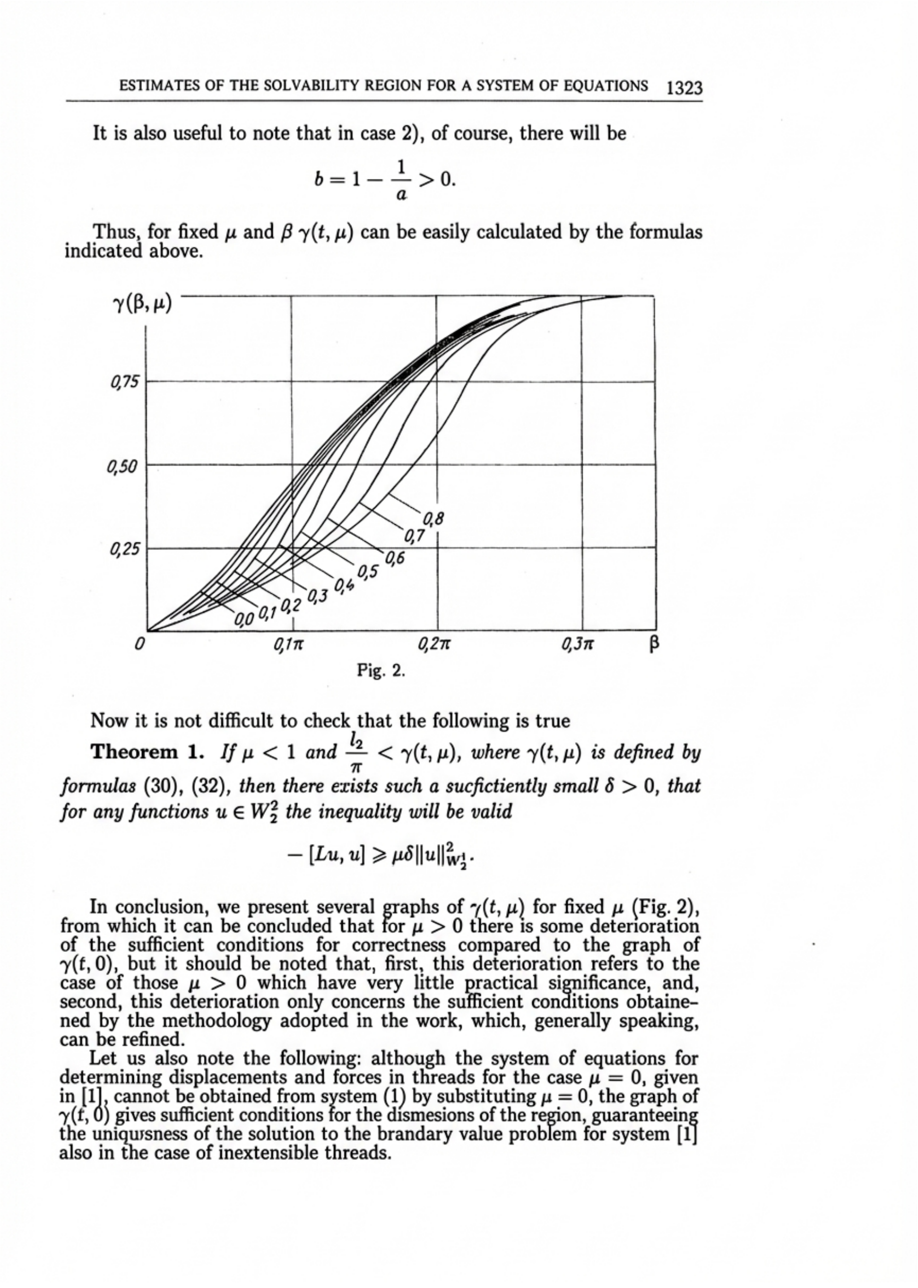
*) The debencance of γ on μ in some cases is omitted for brevity.

Figure 8: Figure 8



It is also useful to note that in case 2), of course, there will be

$$b = 1 - \frac{1}{a} > 0.$$

Thus, for fixed μ and β $\gamma(t, \mu)$ can be easily calculated by the formulas indicated above.

Fig. 2.

Now it is not difficult to check that the following is true

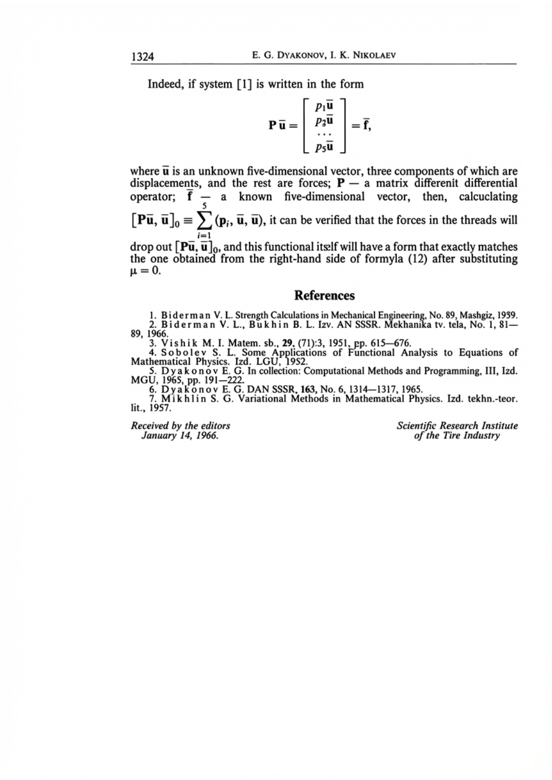
Theorem 1. *If* $\mu < 1$ *and* $\frac{l_2}{\pi} < \gamma(t, \mu)$, *where* $\gamma(t, \mu)$ *is defined by formulas* (30), (32), *then there exists such a sucfictiently small* $\delta > 0$, *that for any functions* $u \in W_2^2$ *the inequality will be valid*

$$-[Lu, u] \geqslant \mu\delta \|u\|^2_{W_2^1}.$$

In conclusion, we present several graphs of $\gamma(t, \mu)$ for fixed μ (Fig. 2), from which it can be concluded that for $\mu > 0$ there is some deterioration of the sufficient conditions for correctness compared to the graph of $\gamma(t, 0)$, but it should be noted that, first, this deterioration refers to the case of those $\mu > 0$ which have very little practical significance, and, second, this deterioration only concerns the sufficient conditions obtainened by the methodology adopted in the work, which, generally speaking, can be refined.

Let us also note the following: although the system of equations for determining displacements and forces in threads for the case $\mu = 0$, given in [1], cannot be obtained from system (1) by substituting $\mu = 0$, the graph of $\gamma(t, 0)$ gives sufficient conditions for the dismesions of the region, guaranteeing the uniqusness of the solution to the brandary value problem for system [1] also in the case of inextensible threads.

Figure 9: Figure 9



Indeed, if system [1] is written in the form

$$\mathbf{P}\,\bar{\mathbf{u}} = \begin{bmatrix} p_1\bar{\mathbf{u}} \\ p_3\bar{\mathbf{u}} \\ \cdots \\ p_5\bar{\mathbf{u}} \end{bmatrix} = \bar{\mathbf{f}},$$

where $\bar{\mathbf{u}}$ is an unknown five-dimensional vector, three components of which are displacements, and the rest are forces; $\mathbf{P}$ — a matrix differenit differential operator; $\bar{\mathbf{f}}$ — a known five-dimensional vector, then, calcuclating $[\mathbf{P}\bar{\mathbf{u}}, \bar{\mathbf{u}}]_0 \equiv \sum_{i=1}^{5} (\mathbf{p}_i, \bar{\mathbf{u}}, \bar{\mathbf{u}})$, it can be verified that the forces in the threads will drop out $[\mathbf{P}\bar{\mathbf{u}}, \bar{\mathbf{u}}]_0$, and this functional itself will have a form that exactly matches the one obtained from the right-hand side of formyla (12) after substituting $\mu = 0$.

References

1. Biderman V. L. Strength Calculations in Mechanical Engineering, No. 89, Mashgiz, 1959.
2. Biderman V. L., Bukhin B. L. Izv. AN SSSR. Mekhanika tv. tela, No. 1, 81—89, 1966.
3. Vishik M. I. Matem. sb., **29**, (71):3, 1951, pp. 615—676.
4. Sobolev S. L. Some Applications of Functional Analysis to Equations of Mathematical Physics. Izd. LGU, 1952.
5. Dyakonov E. G. In collection: Computational Methods and Programming, III, Izd. MGU, 1965, pp. 191—222.
6. Dyakonov E. G. DAN SSSR, **163**, No. 6, 1314—1317, 1965.
7. Mikhlin S. G. Variational Methods in Mathematical Physics. Izd. tekhn.-teor. lit., 1957.

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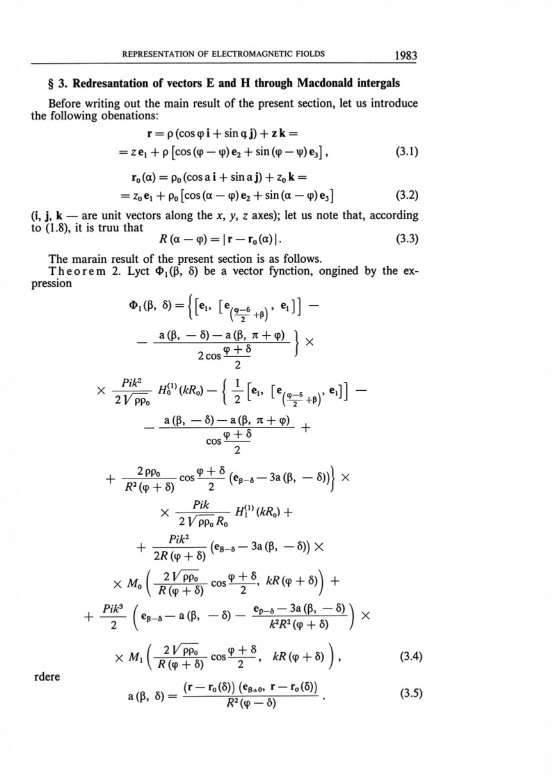
Figure 10: Figure 10



REPRESENTATION OF ELECTROMAGNETIC FIOLDS 1983

§ 3. Redresantation of vectors E and H through Macdonald intergals

Before writing out the main result of the present section, let us introduce the following obenations:

$$\begin{aligned}\mathbf{r} &= \rho\,(\cos\varphi\,\mathbf{i} + \sin q\,\mathbf{j}) + \mathbf{z}\,\mathbf{k} = \\ &= z\,\mathbf{e}_1 + \rho\left[\cos(\varphi-\varphi)\,\mathbf{e}_2 + \sin(\varphi-\psi)\,\mathbf{e}_3\right],\end{aligned} \tag{3.1}$$

$$\begin{aligned}\mathbf{r}_0(\alpha) &= \rho_0\,(\cos a\,\mathbf{i} + \sin a\,\mathbf{j}) + z_0\,\mathbf{k} = \\ &= z_0\,\mathbf{e}_1 + \rho_0\left[\cos(\alpha-\varphi)\,\mathbf{e}_2 + \sin(\alpha-\varphi)\,\mathbf{e}_3\right]\end{aligned} \tag{3.2}$$

(**i**, **j**, **k** — are unit vectors along the x, y, z axes); let us note that, according to (1.8), it is truu that

$$R(\alpha-\varphi) = |\mathbf{r} - \mathbf{r}_0(\alpha)|. \tag{3.3}$$

The marain result of the present section is as follows.

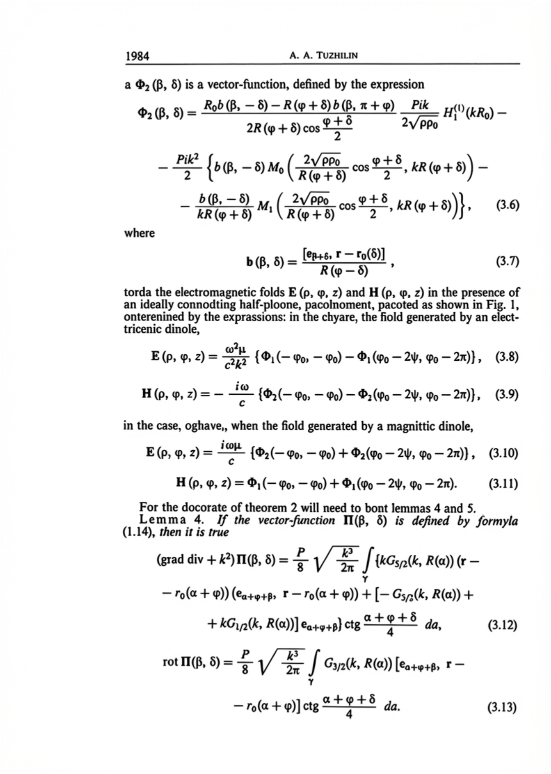
Theorem 2. Lyct $\Phi_1(\beta,\ \delta)$ be a vector fynction, ongined by the expression

$$\begin{aligned}\Phi_1(\beta,\ \delta) = &\left\{\left[\mathbf{e}_1,\ \left[\mathbf{e}_{\left(\frac{\varphi-6}{2}+\beta\right)},\ \mathbf{e}_1\right]\right] - \right. \\ &\left. - \frac{\mathrm{a}(\beta,\ -\delta) - \mathrm{a}(\beta,\ \pi+\varphi)}{2\cos\frac{\varphi+\delta}{2}}\right\} \times \\ &\times \frac{Pik^2}{2\sqrt{\rho\rho_0}}\,H_0^{(1)}(kR_0) - \left\{\frac{1}{2}\left[\mathbf{e}_1,\ \left[\mathbf{e}_{\left(\frac{\varphi-6}{2}+\beta\right)},\ \mathbf{e}_1\right]\right] - \right. \\ &- \frac{\mathrm{a}(\beta,\ -\delta) - \mathrm{a}(\beta,\ \pi+\varphi)}{\cos\frac{\varphi+\delta}{2}} + \\ &\left. + \frac{2\rho\rho_0}{R^2(\varphi+\delta)}\cos\frac{\varphi+\delta}{2}\left(\mathbf{e}_{\beta-\delta} - 3\mathrm{a}(\beta,\ -\delta)\right)\right\} \times \\ &\times \frac{Pik}{2\sqrt{\rho\rho_0}\,R_0}\,H_1^{(1)}(kR_0) + \\ &+ \frac{Pik^2}{2R(\varphi+\delta)}\left(\mathbf{e}_{8-\delta} - 3\mathrm{a}(\beta,\ -\delta)\right) \times \\ &\times M_0\left(\frac{2\sqrt{\rho\rho_0}}{R(\varphi+\delta)}\cos\frac{\varphi+\delta}{2},\ kR(\varphi+\delta)\right) + \\ &+ \frac{Pik^3}{2}\left(\mathbf{e}_{\beta-\delta} - \mathrm{a}(\beta,\ -\delta) - \frac{\mathbf{e}_{D-\delta} - 3\mathrm{a}(\beta,\ -\delta)}{k^2R^2(\varphi+\delta)}\right) \times \\ &\times M_1\left(\frac{2\sqrt{\rho\rho_0}}{R(\varphi+\delta)}\cos\frac{\varphi+\delta}{2},\ \ kR(\varphi+\delta)\right),\end{aligned} \tag{3.4}$$

rdere

$$\mathrm{a}(\beta,\ \delta) = \frac{(\mathbf{r} - \mathbf{r}_0(\delta))\,(\mathbf{e}_{B\Delta 0},\ \mathbf{r} - \mathbf{r}_0(\delta))}{R^2(\varphi-\delta)}. \tag{3.5}$$

Figure 11: Figure 11



a $\mathbf{\Phi}_2(\beta, \delta)$ is a vector-function, defined by the expression

$$\mathbf{\Phi}_2(\beta, \delta) = \frac{R_0 b(\beta, -\delta) - R(\varphi+\delta)\, b(\beta, \pi+\varphi)}{2R(\varphi+\delta)\cos\frac{\varphi+\delta}{2}} \frac{Pik}{2\sqrt{\rho\rho_0}} H_1^{(1)}(kR_0) -$$

$$- \frac{Pik^2}{2}\left\{ b(\beta, -\delta) M_0\left(\frac{2\sqrt{\rho\rho_0}}{R(\varphi+\delta)}\cos\frac{\varphi+\delta}{2}, kR(\varphi+\delta)\right) -\right.$$

$$\left.- \frac{b(\beta, -\delta)}{kR(\varphi+\delta)} M_1\left(\frac{2\sqrt{\rho\rho_0}}{R(\varphi+\delta)}\cos\frac{\varphi+\delta}{2}, kR(\varphi+\delta)\right)\right\}, \quad (3.6)$$

where

$$\mathbf{b}(\beta, \delta) = \frac{[\mathbf{e}_{\beta+\delta}, \mathbf{r} - \mathbf{r}_0(\delta)]}{R(\varphi-\delta)}, \quad (3.7)$$

torda the electromagnetic folds $\mathbf{E}(\rho, \varphi, z)$ and $\mathbf{H}(\rho, \varphi, z)$ in the presence of an ideally connodting half-ploone, pacolnoment, pacoted as shown in Fig. 1, onterenined by the exprassions: in the chyare, the fiold generated by an electricenic dinole,

$$\mathbf{E}(\rho, \varphi, z) = \frac{\omega^2\mu}{c^2k^2}\{\mathbf{\Phi}_1(-\varphi_0, -\varphi_0) - \mathbf{\Phi}_1(\varphi_0 - 2\psi, \varphi_0 - 2\pi)\}, \quad (3.8)$$

$$\mathbf{H}(\rho, \varphi, z) = -\frac{i\omega}{c}\{\mathbf{\Phi}_2(-\varphi_0, -\varphi_0) - \mathbf{\Phi}_2(\varphi_0 - 2\psi, \varphi_0 - 2\pi)\}, \quad (3.9)$$

in the case, oghave,, when the fiold generated by a magnittic dinole,

$$\mathbf{E}(\rho, \varphi, z) = \frac{i\omega\mu}{c}\{\mathbf{\Phi}_2(-\varphi_0, -\varphi_0) + \mathbf{\Phi}_2(\varphi_0 - 2\psi, \varphi_0 - 2\pi)\}, \quad (3.10)$$

$$\mathbf{H}(\rho, \varphi, z) = \mathbf{\Phi}_1(-\varphi_0, -\varphi_0) + \mathbf{\Phi}_1(\varphi_0 - 2\psi, \varphi_0 - 2\pi). \quad (3.11)$$

For the docorate of theorem 2 will need to bont lemmas 4 and 5.

Lemma 4. *If the vector-function* $\mathbf{\Pi}(\beta, \delta)$ *is defined by formyla* (1.14), *then it is true*

$$(\operatorname{grad}\operatorname{div} + k^2)\mathbf{\Pi}(\beta, \delta) = \frac{P}{8}\sqrt{\frac{k^3}{2\pi}}\int_\gamma \{kG_{5/2}(k, R(\alpha))(\mathbf{r} -$$

$$- r_0(\alpha+\varphi))(\mathbf{e}_{a+\varphi+\beta}, \mathbf{r} - r_0(\alpha+\varphi)) + [-G_{5/2}(k, R(\alpha)) +$$

$$+ kG_{1/2}(k, R(\alpha))]\,\mathbf{e}_{a+\varphi+\beta}\}\operatorname{ctg}\frac{\alpha+\varphi+\delta}{4}\, da, \quad (3.12)$$

$$\operatorname{rot}\mathbf{\Pi}(\beta, \delta) = \frac{P}{8}\sqrt{\frac{k^3}{2\pi}}\int_\gamma G_{3/2}(k, R(\alpha))\,[\mathbf{e}_{a+\varphi+\beta}, \mathbf{r} -$$

$$- r_0(\alpha+\varphi)]\operatorname{ctg}\frac{\alpha+\varphi+\delta}{4}\, da. \quad (3.13)$$

Figure 12: Figure 12