ON NONLINEAR OSCILLATIONS OF THIN PLATES WITH ACCOUNT OF ROTATIONAL INERTIA

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Abstract

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ON NONLINEAR OSCILLATIONS OF THIN PLATES WITH ACCOUNT OF ROTATIONAL INERTIA

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The problem of nonlinear oscillations of a plate was studied in works ([1−4]), where the system of equations

$$\rho h w_{tt} + D\Delta^2 w = Z(x,y,t) + F_{xx}w_{yy} + F_{yy}w_{xx} - 2F_{xy}w_{xy},$$
$$\Delta^2 F = hE\left[w_{xy}^2 - w_{xx}w_{yy}\right]^* . \quad (1)$$

The contour Γ bounds the domain Ω of the variables x and y. The differential equations (1) are non-wave equations: their integration leads to instantaneous propagation of disturbances, which contradicts the general dynamical equations of the theory of elasticity. It should also be noted that the study of equations (1) is connected with substantial mathematical difficulties, for example, in the question of uniqueness of the solution.

In the present work the same problem will be studied with account of the rotational inertia of the plate elements. Then we obtain the system of equations (see ([5,6]))

$$D\Delta^2 w - \gamma^2 \Delta w_{tt} + w_{tt} = Z(x,y,t) + F_{xx}w_{yy} + F_{yy}w_{xx} - 2F_{xy}w_{xy},$$
$$\Delta^2 F = hE\left[w_{xy}^2 - w_{xx}w_{yy}\right] . \quad (2)$$

Consider equations (2) with boundary conditions (3) and (4)

$$w|_\Gamma = \partial w/\partial \nu|_\Gamma = 0; \quad (3)$$

$$F|_\Gamma = \partial F/\partial \nu|_\Gamma = 0 \quad (4)$$

and initial conditions

$$w|_{t=0} = w_t|_{t=0} = 0. \tag{5}$$

In the proposed work it will be proved: the existence of a generalized solution of problem (2)–(5) in a certain energy space; the uniqueness of the solution of problem (2)–(5) in the same space; with account of dissipative terms, a qualitative study of the solution for "large" time will be carried out.

Let H be the space obtained by completing, in the norm (6), the set of smooth functions defined in the cylinder $Q = [0, T] \times \Omega$ and satisfying conditions (3):

$$\|w\|_H^2 = \int_0^T \int_\Omega \left[w_t^2 + \gamma^2 \operatorname{grad}^2 w_t + D(\Delta w)^2\right] d\Omega\, dt. \tag{6}$$

Analogously to [1,7], we introduce the concept of a generalized solution of problem (2)–(5) in the space H.

Consider the auxiliary problem

$$D\Delta^2 w - \gamma^2 \Delta w_{tt} + w_{tt} = f(x, y, t); \tag{7}$$

$$w|_\Gamma = \partial w/\partial \nu|_\Gamma = 0; \tag{8}$$

$$w|_{t=0} = w_t|_{t=0} = 0. \tag{9}$$

* In what follows we shall assume $\rho h = 1$.

It is easy to show, following (1), that for the generalized solution of problem (7)–(9) the equality

$$\int_\Omega D(\Delta w)^2\, d\Omega + \gamma^2 \int_\Omega \operatorname{grad}^2 w_t\, d\Omega + \int_\Omega w_t^2\, d\Omega = 2 \int_0^T \int_\Omega w_t f\, d\Omega\, dt. \tag{10}$$

is valid.

We now seek Bubnov-Galerkin approximations to the generalized solution of problem (2)–(5) in the space H. We choose $w^{(n)}$ in the form

$$w^{(n)} = \sum_{k=1}^{n} q_k^{(n)}(t) \psi_k(x, y). \tag{11}$$

Here ψ_k are the eigenfunctions of the biharmonic operator under conditions (3), and $F^{(n)}$ is determined from the equation

$$\Delta^2 F^{(n)} = hE\left[(w_{xy}^{(n)})^2 - w_{xx}^{(n)} w_{yy}^{(n)}\right] \tag{12}$$

and the boundary conditions (4)

$$F^{(n)}\big|_\Gamma = \partial F^{(n)}/\partial\nu\big|_\Gamma = 0.$$

Then for $q_k^{(n)}$ we obtain the system of ordinary differential equations

$$\ddot{q}_k^{(n)} + \sum_{i=1}^{n} c_{ik}\ddot{q}_i^{(n)} + \frac{\partial\Phi_n\left(q_1^{(n)}, q_2^{(n)}, \dots, q_n^{(n)}\right)}{\partial q_k^{(n)}} = Z_k(t). \tag{13}$$

Here

$$Z_k = \int_\Omega Z(x,y,t)\psi_k\, d\Omega; \qquad c_{ik} = -\int_\Omega \Delta\psi_i \cdot \psi_k\, d\Omega;$$

Φ_n is a positive definite functional

$$\Phi_n = \frac{1}{2}\int_\Omega (\Delta w^{(n)})^2\, d\Omega + \frac{1}{2}\int_\Omega (\Delta F^{(n)})^2\, d\Omega. \tag{14}$$

By virtue of (13) and (14) we obtain the a priori equality

$$D\int_\Omega [\Delta w^{(n)}]^2\, d\Omega + \gamma^2 \int_\Omega \left[(w_{xt}^{(n)})^2 + (w_{yt}^{(n)})^2\right]\, d\Omega + \int_\Omega (w_t^{(n)})^2\, d\Omega +$$

$$+\frac{1}{Eh}\int_\Omega [\Delta F^{(n)}]^2\, d\Omega = 2\int_0^T \int_\Omega Z w_t^{(n)}\, d\Omega\, dt \tag{15}$$

and, as a consequence of (15), the a priori estimate

$$\|w^{(n)}\|_H < C.$$

It follows from this that the set of approximate solutions $w^{(n)}$ is weakly compact in H, and every weak limit of $w^{(n)}$ in H is a generalized solution of problem (2)–(5).

We now show that the generalized solution of problem (2)–(5) is unique in the space H. Let w_1 and w_2 be two distinct solutions from the space H. Then $v = w_1 - w_2$ solves problem (7)–(9) with right-hand side f

$$f \equiv F_{1xx}w_{1yy} + F_{1yy}w_{1xx} - 2F_{1xy}w_{1xy} - F_{2xx}w_{2yy} - F_{2yy}w_{2xx} + 2F_{2xy}w_{2xy}.$$

By virtue of (10) we obtain the inequality

$$D\int_\Omega (\Delta v)^2\,d\Omega + \gamma^2\int_\Omega (v_{xt}^2 + v_{yt}^2)\,d\Omega + \int_\Omega v_t^2\,d\Omega = 2\int_0^T\int_\Omega v_t f\,d\Omega\,dt =$$

$$= 2\int_0^T\int_\Omega \{v_{tx}\left[w_{1y}F_{1xy} - w_{2y}F_{2xy} - w_{1x}F_{1yy} + w_{2x}F_{2yy}\right] +$$

$$+ v_{ty}\left[w_{1x}F_{1xy} - w_{2x}F_{2xy} - w_{1y}F_{1xx} + w_{2y}F_{2xx}\right]\}\,d\Omega\,dt \le$$

$$\ll \frac{2}{\gamma}\|v\|_H\|v_y\|_{L_4Q}\|F_{1xy}\|_{L_4Q} +$$

$$+\frac{2}{\gamma}\|v\|_H\|w_{2y}\|_{L_4Q}\|F_{1xy} - F_{2xy}\|_{L_4Q} + \cdots \ll C\|v\|_H^2. \quad (16)$$

In deriving inequality (16), the embedding theorems, the properties of equation (2), and also the fact that the solutions w_1 and w_2 belong to the space H were used essentially.

Integrating inequality (16) from 0 to T, we obtain

$$\|v\|_H^2 \ll CT\|v\|_H^2.$$

Hence it follows that $v \equiv 0$, and the generalized solution in the space H is unique.

If the effect of damping is taken into account, then system (2) is transformed into the form

$$D\Delta^2 w - \gamma^2\Delta w_{tt} + w_{tt} + \varepsilon_1 w_t - \varepsilon_2\Delta w_t = Z + F_{xx}w_{yy} + F_{yy}w_{xx} - 2F_{xy}w_{xy}; \quad (17a)$$

$$\Delta^2 F = hE[w_{xy}^2 - w_{xx}w_{yy}]; \quad (17b)$$

$$w|_\Gamma = \partial w/\partial\nu|_\Gamma = F|_\Gamma = \partial F/\partial\nu|_\Gamma = 0. \quad (17c)$$

Introduce the notation

$$\|w\|^2_{H\Omega} = \int_\Omega D(\Delta w)^2\, d\Omega + \int_\Omega \gamma^2 \operatorname{grad}^2 w_t\, d\Omega + \int_\Omega w_t^2\, d\Omega; \tag{18}$$

$$\|w\|^2_{HQ_\tau} = \int_0^\tau \|w\|^2_{H\Omega}\, dt; \tag{19}$$

$$\|w\|_{\hat{H}Q_\tau} = \max_{0\le t\le\tau} \|w\|_{H\Omega}. \tag{20}$$

The existence and uniqueness theorems in the space H_{Q_τ} are established for system (17) analogously to what was set forth for system (2). Equalities of the type (10) and (15) ensure that the solution belongs to the space $\hat{H}_{Q_\tau}$.

Let us note that the Bubnov–Galerkin system for problem (17) has the form

$$\ddot{q}_k^{(n)} + \varepsilon_1 \dot{q}_k^{(n)} + \sum_{i=1}^n c_{ik}\ddot{q}_i^{(n)} + \varepsilon_2 \sum_{i=1}^n c_{ik}\dot{q}_i^{(n)} + \frac{\partial \Phi_n(q_1^{(n)}, \dots q_m^{(n)})}{\partial q_k^{(n)}} = Z_k(t) \tag{21}$$

(see formulas (13) and (14)).

We now apply methods of ordinary differential equations (see ([8])) for a further investigation of system (17). Consider the function V_n

$$V_n = \frac{1}{2}\sum_{k=1}^n \dot{q}_k^{(n)2} + \Phi_n = \frac{1}{2}\sum_{i,k=1}^n c_{ki}\dot{q}_k^{(n)}\dot{q}_i^{(n)} +$$
$$+ c_0^2 \left[\frac{1}{2}\sum_{k=1}^n q_k^{(n)2} + \frac{1}{\varepsilon_1}\sum_{i,k=1}^n c_{ik}\dot{q}_k^{(n)} q_i^{(n)} + \right.$$
$$\left. + \frac{1}{\varepsilon_1}\sum_{k=1}^n \dot{q}_k^{(n)} q_k^{(n)} + \frac{\varepsilon_2}{2\varepsilon_1}\sum_{i,k=1}^n c_{ik} q_i^{(n)} q_k^{(n)} \right]. \tag{22}$$

Here c_0 is a suitably chosen constant, $c_0 = c_0(\varepsilon_1, \varepsilon_2)$. For V_n the estimates

$$c_1\|w^{(n)}\|^2_{H\Omega} \ge V_n \ge c_2\|w^{(n)}\|^2_{H\Omega}.$$

Differentiate V_n with respect to t; by virtue of equations (21)

$$\frac{dV_n}{dt} = -\varepsilon_1 \sum_{k=1}^{n} \dot{q}_k^{(n)2} - \varepsilon_2 \sum_{ik=1}^{n} c_{ik} q_i^{(n)} \dot{q}_k^{(n)} + \\ + c_0^2 \left[-\sum_{k=1}^{n} q_k^{(n)} \frac{\partial \Phi_n}{\partial q_k^{(n)}} + \frac{1}{\varepsilon_1} \sum_{k=1}^{n} \dot{q}_k^{(n)2} + \frac{1}{\varepsilon_1} \sum_{i,k=1}^{n} c_{ik} q_i^{(n)} \dot{q}_k^{(n)} \right] + \\ + \sum_{k=1}^{n} \dot{q}_k^{(n)} Z_k + c_0^2 \sum_{k=1}^{n} q_k^{(n)} Z_k \leq -a_0^2 V_n + b_0^2 Z_0^2.$$

Here

$$Z_0^2 = \max_{0 \leq t \leq +\infty} \int_\Omega Z^2(x, y, t)\, d\Omega.$$

By virtue of (23) and (24), the function V_n is a Lyapunov-type function (see (9)) for the system (21). With the aid of V_n it is established that the system (21), and consequently (17), is generalized dissipative, namely: let φ_1 and φ_2 be admissible initial conditions for the system (17),

$$w\big|_{t=0} = \varphi_1, \qquad w_t\big|_{t=0} = \varphi_2;$$

let w be a solution of system (17) under the indicated initial conditions. Then for all admissible φ_1 and φ_2 one can indicate an $R > 0$ such that to each admissible pair φ_1, φ_2 there corresponds a t_0, beginning with which the inequality $\|w\|_{H_\Omega}^2 \leq R^2$ holds for almost all $t \in [t_0, +\infty)$. We shall call the initial conditions admissible if $\operatorname{grad} \varphi_2 \in L_{2\Omega}$, $\Delta\varphi_1 \in L_{2\Omega}$. From the generalized dissipativity of system (17) it follows:

Theorem. *Under a periodic load $Z(x, y, t+T) = Z(x, y, t)$, problem (17) has in the space $\hat{H}_{\Omega T}$ at least one periodic solution, and this solution can be found by the Bubnov-Galerkin method.*

Remark. In the case of a circular symmetrically loaded plate under a periodic load $Z(x, y, t+T) = Z(x, y, t)$, the system (17) is convergent for sufficiently large ε ($\varepsilon = \min(\varepsilon_1, \varepsilon_2)$), namely:

1. For all admissible initial conditions, in the space $\hat{H}_{Q\infty}$ there exists a generalized solution of problem (17).
2. Equations (17) have a unique T-periodic generalized solution $w_0(t) \in \hat{H}_{QT}$.
3. This solution is stable in the sense of Lyapunov in the metric $\hat{H}_{Q\infty}$.
4. For any solution of problem (17) $w(t) \in \hat{H}_{Q\infty}$, the limiting relation

$$\lim_{t\to+\infty} \|w(t) - w_0(t)\|_{H_\Omega} = 0$$

holds.

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Note: Figure translations are in progress. See original paper for figures.