\(A\)-SETS IN COMPLETE METRIC SPACES

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Abstract

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A-SETS IN COMPLETE METRIC SPACES

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As is known, every A-set in a complete metric space with a countable base is either at most countable, or contains a topological image of the Cantor perfect set (the Alexandrov-Hausdorff theorem ([1],[2])).

Stone ([3]) proved the following theorem: every Borel set in a complete metric space (not necessarily separable) is either σ-discrete, or contains a Cantor perfect set.

We prove the following proposition:

Main theorem. *An A-set in a complete metric space (not necessarily separable) is either σ-discrete, or contains a Cantor perfect set.*

Remark 1. From this theorem there follow both the Alexandrov-Hausdorff theorem and Stone’ s theorem.

Before proving our theorem, we introduce several necessary definitions and formulate a number of simple auxiliary propositions.

Definition 1. Let A be some set lying in a metric space X. A is called **relatively discrete** if every point $x \in A$ is an isolated point of the set A; A is called **discrete** if A is relatively discrete and closed in X; A is called **metrically discrete** if there exists such an $\varepsilon < 0$ that for any two points x and y of A, $x \neq y$, one necessarily has $\rho(x, y) \geq \varepsilon$.

Lemma 1 (see ([3])). *The following assertions concerning a subset A of a metric space X are equivalent:*

1) $A = \bigcup\limits_{i=1}^{\infty} A_i$, where A_i is relatively discrete, $i = 1, 2, ...$

2) $A = \bigcup\limits_{i=1}^{\infty} B_i$, where B_i is discrete, $i = 1, 2, ...$

3) $A = \bigcup\limits_{i=1}^{\infty} C_i$, where C_i is metrically discrete, $i = 1, 2, ...$

Definition 2. A subset A of a metric space X is called **σ-discrete** if A can be represented in at least one of the forms 1), 2), or 3). The empty set is always

σ-discrete.

Remark 2. 1°. The property of a set A of being σ-discrete is topologically invariant.

2°. Every subset of a σ-discrete set is itself σ-discrete.

3°. A σ-discrete space of weight τ has cardinality $\leq \tau$. In particular, in a space of countable weight every σ-discrete set is at most countable.

It follows from this that the following properties of a set A are mutually exclusive: a) A is σ-discrete; b) A contains a Cantor perfect set.

4°. If a set A is not σ-discrete, then it is uncountable.

5°. The sum of a finite or countable number of σ-discrete sets is a σ-discrete set.

Definition 3. A point $x \in A \subseteq X$ is called a **point of local σ-discreteness of the set** A if there exists a neighborhood Ox of this point x in X for which the set $Ox \cap A$ is necessarily σ-discrete. A point of local σ-discreteness of the set A will also be called a **σ-discrete point of the set** A. A point of the set A that is not a σ-discrete point of this set will be called a **non-σ-discrete point of the set** A. The set of all points of local σ-discreteness of the set A will be denoted by A_σ; the set of non-σ-discrete points of the set A will be denoted by $A_{\bar{\sigma}}$. Obviously,

$$A = A_\sigma \cup A_{\bar{\sigma}}; \quad A_\sigma \cap A_{\bar{\sigma}} = \Lambda.$$

Definition 4. A space X is called **locally σ-discrete** if each of its points is a σ-discrete point of this space, i.e. if $X_\sigma = \Lambda$. Obviously, a σ-discrete set is locally σ-discrete.

Lemma 2 (see ([3])). *A locally σ-discrete paracompact space is necessarily σ-discrete.*

Remark 3. 1°. For every set A in a metric space X, the set A_σ is always σ-discrete.

2°. If A is not σ-discrete, then $A_{\bar{\sigma}}$ is also not σ-discrete. If, however, A is σ-discrete, then, obviously, $A_{\bar{\sigma}} = \Lambda$. Thus either $A_{\bar{\sigma}} = \Lambda$, or $A_{\bar{\sigma}}$ is not σ-discrete (and hence is uncountable).

Lemma 3. *Let G be an open set of the space X, and let $A \subseteq X$. If the set $M = G \cap A_{\bar{\sigma}}$ is nonempty, then M is not σ-discrete, and moreover $M_\sigma = \Lambda$.*

Proof of Lemma 3. Let $M \neq \Lambda$. Suppose the contrary, namely that M is σ-discrete. Let $x \in M$. $M = G \cap A_{\bar{\sigma}}$. Then G is a neighborhood of the point x in X; the set

$$G \cap A = (G \cap A_\sigma) \cup M$$

is σ-discrete, which contradicts the fact that $x \in A_{\bar{\sigma}}$. Thus the set M is not σ-discrete.

We now prove that $M_\sigma = \Lambda$. Suppose the contrary, and let $M_\sigma \neq \Lambda$. Take an arbitrary point $y \in M_\sigma$. By the definition of the set M_σ, there exists a neighborhood Oy of the point y in X for which the set $Oy \cap M$ is σ-discrete. Since $y \in G$, $O_{1y} = G \cap Oy$ is a neighborhood of the point y in X. Then

$$O_{1y} \cap A = (O_{1y} \cap A_\sigma) \cup (G \cap Oy \cap A_{\bar{\sigma}}) = (O_{1y} \cap A_\sigma) \cup (M \cap Oy).$$

It follows from this that $O_{1y} \cap A$ is necessarily σ-discrete, which contradicts the fact that $y \in A_{\bar{\sigma}}$. Consequently, $M_\sigma = \Lambda$. The lemma is proved.

Proof of the main theorem. In view of Remark 2, 3°, the formulation of the theorem is correct. The proof of this theorem repeats the arguments of Alexandrov—Hausdorff in the proof of the theorem on the cardinality of A-sets lying in complete spaces with a countable base (see ([1],[2])). The difference is that, in constructing spherical neighborhoods, we shall use not condensation points but non-σ-discrete points. Thus, let A be an A-set lying in a complete metric space X, i.e.

$$A = \bigcup_{(i,k,l,\dots)} (F_i \cap F_{ik} \cap F_{ikl} \cap \cdots),$$

where the summation extends over all possible sequences $(i, k, l, \dots)$ of natural numbers, and all the sets $F_{ikl\dots s}$ are closed in X. Obviously, one may assume that for each numerical sequence $(i, k, l, \dots)$ the inclusions

$$F_i \supseteq F_{ik} \supseteq F_{ikl} \supseteq \cdots$$

hold.

If we fix the first index, we obtain the sets

$$A_i = \bigcup_{(k,l,\dots)} (F_i \cap F_{ik} \cap F_{ikl} \cap \dots), \qquad A_i \subseteq F_i, \qquad i = 1, 2, \dots, \qquad A = \bigcup_{i=1}^{\infty} A_i.$$

When two indices are fixed, we obtain the sets

$$A_{ik} = \bigcup_{(l,\dots)} (F_i \cap F_{ik} \cap F_{ikl} \cap \dots), \qquad A_{ik} \subseteq F_{ik}, \qquad i = 1, 2, \dots; \; k = 1, 2, \dots,$$

$$A_i = \bigcup_{k=1}^{\infty} A_{ik}$$

and so on.

Let A not be σ-discrete. We must prove that then it contains a Cantor perfect set. By virtue of remark 3, 2^0, the set $A_{\bar{\sigma}}$ is not σ-discrete. Therefore (see remark 2, 4^0) in this set one can choose two distinct points a_1 and a_2. Surround them by disjoint closed balls V_1 and V_2. The corresponding (i.e. having the same center and radius) open balls will be denoted by U_1 and U_2. Let $(p, q, r, \ldots)$ be a binary numerical sequence composed of the digits 1 or 2.

Consider the sets A_i, $A = \bigcup_{i=1}^{\infty} A_i$. Since A is not σ-discrete, there exist two natural numbers i_1 and i_2 (not necessarily distinct) such that A_{i_p} is not σ-discrete. At the same time, we may assume that i_p has been chosen in such a way that

$$A_{i_p\bar{\sigma}} \cap U_p \neq \Lambda.$$

We prove this. Suppose the contrary; let, for example, for $p = 1$ and for every $i = 1, 2, \ldots$ one have

$$A_{i\bar{\sigma}} \cap U_1 = \Lambda.$$

Then

$$A \cap U_1 = \bigcup_{i=1}^{\infty} (A_i \cap U_1) = \bigcup_{i=1}^{\infty} (A_{i\sigma} \cap U_1),$$

and, consequently, $A \cap U_1$ is σ-discrete (see remarks 3, 1^0; 2, 2^0; 2, 5^0). But this means that the point $a_1 \in A \cap U_1$ is a σ-discrete point of the set A (by definition 3), which contradicts the fact that $a_1 \in A_{\bar{\sigma}}$. Thus, we may assume that i_p, moreover, has been chosen in such a way that $A_{i_p\bar{\sigma}} \cap U_p \neq \Lambda$.

But if $A_{i_p\bar{\sigma}} \cap U_p \neq \Lambda$, then, by lemma 3, this set is not σ-discrete. Therefore in this set one can choose two distinct points a_{p1} and a_{p2}. Surround them by disjoint closed balls V_{p1} and V_{p2}, $V_{pq} \subseteq U_p$, q equal to 1 or 2. The corresponding open balls will be denoted by U_{p1} and U_{p2}.

Now consider the sets $A_{i_p k}$. We have

$$A_{i_p} = \bigcup_{k=1}^{\infty} A_{i_p k}.$$

Since A_{i_p} is not σ-discrete, there exist two natural numbers k_1 and k_2 (not necessarily distinct) for which the set $A_{i_p k_q}$ is not σ-discrete. At the same time, we may assume that k_q has been chosen in such a way that

$$A_{i_p k_q \bar{\sigma}} \cap U_{pq} \neq \Lambda.$$

We prove this. Suppose the contrary; let, for example, for $q = 1$ and for every $k = 1, 2, \ldots$ one have

$$A_{i_p k\bar{\sigma}} \cap U_{p1} = \Lambda.$$

Then

$$A_{i_p} \cap U_{p1} = \bigcup_{k=1}^{\infty} (A_{i_p k} \cap U_{p1}) = \bigcup_{k=1}^{\infty} (A_{i_p k\sigma} \cap U_{p1}),$$

and, consequently, the set $A_{i_p} \cap U_{p1}$ is necessarily σ-discrete, whence it follows that $a_{p1} \in A_{i_p \sigma}$, which cannot be, since $a_{p1} \in A_{i_p \bar{\sigma}}$. Thus, k_q can be chosen in such a way that

$$A_{i_p k_q \bar{\sigma}} \cap U_{pq} \neq \Lambda,$$

and $A_{i_p k_q}$ is not σ-discrete. But if

$$A_{i_p k_q \bar{\sigma}} \cap U_{pq} \neq \Lambda,$$

then by lemma 3 this set is not σ-discrete. Therefore in this set one can choose two distinct points a_{pq1} and a_{pq2}, and so on. Thus,

to each binary sequence $(p, q, r, \ldots)$ there corresponds a sequence of natural numbers $(i_p, k_q, l_r, \ldots)$ such that all the sets $A_{i_p} \cap U_p$, $A_{i_p k_q} \cap U_{pq}$, $A_{i_p k_q l_r} \cap U_{pqr}, \ldots$ are nonempty (indeed, they are even uncountable). But then the sets

$$F_{i_p} \cap V_p, \quad F_{i_p k_q} \cap V_{pq}, \quad F_{i_p k_q l_r} \cap V_{pqr}, \ldots$$

will also be nonempty.

If the radii of the balls $V_p, V_{pq}, V_{pqr}, \ldots$ are chosen respectively less than $1,\ 1/2,\ 1/3, \ldots$, then the latter sets form a sequence of nonempty decreasing closed sets with diameters tending to zero, and, consequently, their intersection in the complete metric space is nonempty and consists of a single point x, which belongs both to the set A and to the Cantor perfect set

$$D = \bigcup_{(p,q,r,\ldots)} (V_p \cap V_{pq} \cap V_{pqr} \cap \ldots).$$

Since this holds for every binary sequence $(p, q, r, \ldots)$, every point $x \in D$ is contained also in the set A. Thus, $D \subseteq A$, as was required to prove.

Additional properties of the set A_σ.

1°. A_σ is closed in A.

2°. A_σ is dense in itself.

3°. From 1° and 2° it follows that A_σ is perfect in A.

4°. $A_{\sigma\sigma} = \Lambda$.

5°. Every point $x \in A_\sigma$ is a condensation point of this set (and hence of the set A).

The following simple result holds.

Theorem. A scattered set is σ-discrete.

Remark 4. A set is called scattered if it contains no (nonempty) dense-in-itself subset.

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Note: Figure translations are in progress. See original paper for figures.