ON CHARACTERISTIC PROPERTIES OF SEISMIC TRAVEL-TIME CURVES

GEOPHYSICS

1967

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Fig. 1

Figure 1: Fig. 1

Abstract

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UDC 550.34

GEOPHYSICS

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ON CHARACTERISTIC PROPERTIES OF SEISMIC TRAVEL-TIME CURVES

(Presented by Academician M. A. Sadovsky, September 19, 1966)

Investigating the inverse problem of geometrical seismics for a spherically symmetric medium with a finite number of waveguides, in papers ([1]) and (in more detail) ([2]) we considered questions of uniqueness in determining a velocity section from travel-time curves of surface and deep sources. In this note we shall formulate conditions under which a solution of the inverse problem exists.

I. 1. Let us recall the formulation of the problem (see Fig. 1a). At a point A on the circumference, at the moment $t = 0$, an impulsive disturbance arises, which propagates inside the disk C along rays according to the laws of geometrical optics with velocity $v(r)$. At every point where the impulse again emerges on the circumference, the arrival time t is recorded.

Fig. 1

Denote by $\tilde{\theta}$ the angular epicentral distance, and by α the angle between the ray and the radius at the source. Then in the plane θ, t we shall have the curve $\tilde{\Gamma}_0\{\theta = \tilde{\theta}(\alpha),\ t = t(\alpha)\}$. It is required to determine from the curve $\tilde{\Gamma}_0$ the propagation velocity of the impulse $v(r)$, $r \in [0, R]$.

2. Assuming that not only the angle $\tilde{\theta}$ is known, but also the number k of revolutions of the ray about the center of the disk, we consider the curve $\Gamma_0\{\theta = \Theta(\alpha),\ t = t(\alpha)\}$, $\alpha \in (0, \pi/2)$, where $\Theta(\alpha) = \widetilde{\Theta}(\alpha) + 2\pi k$.
3. The transformation

$$x = \frac{R\theta}{v(R)}, \qquad y = \frac{R}{v(R)} \ln \frac{R}{r}, \qquad u(y) = \frac{v\left(Re^{-v(R)y/R}\right)}{v(R)e^{-v(R)y/R}}$$

makes it possible to consider a simpler problem of the propagation of an impulse

Fig. 2

Figure 2: Fig. 2

in the half-plane x, y, $y \geq 0$, with velocity $u(y)$, where $u(0) = 1$. The curves $\tilde{\Gamma}_0$ and Γ_0 are then transformed into curves in the x, t plane

$$\tilde{\Gamma}\{x = 2\widetilde{X}(p),\ t = 2T(p)\}, \qquad \Gamma\{x = 2X(p),\ t = 2T(p)\}, \qquad p \in (0, 1),$$

where $p = \sin\alpha$ is the ray parameter; $X(p)$ is the abscissa of the deepest point of the ray with parameter p; $T(p)$ is the time of motion to the deepest point; $\widetilde{X}(p) \equiv X(p) \pmod{\pi R/v(R)}$, $0 \leq \widetilde{X}(p) < \pi R/v(R)$ (see Fig. 1b).

4. We shall not consider the problem of determining $X(p)$ and $T(p)$ from Γ (see ([2])) and of the mapping $\Gamma \to \tilde{\Gamma}$, but shall deal with questions of existence in determining $u(y)$ from $X(p)$ and $T(p)$.

II. 1. The functions $X(p)$ and $T(p)$ are expressed in terms of $u(y)$ as follows:

$$X(p) = \int_0^{Y(p)} \frac{pu(y)\,dy}{\sqrt{1 - p^2u^2(y)}}, \qquad T(p) = \int_0^{Y(p)} \frac{dy}{u(y)\sqrt{1 - p^2u^2(y)}}, \qquad p \in (0, 1), \tag{1}$$

where $Y(p) = \inf\{y,\ pu(y) \geqslant 1\}$ is the ordinate of the deepest point of the ray with parameter p.

2. With respect to $u(y)$ we shall assume that it is a positive, piecewise twice continuously differentiable function, bounded on every segment of the half-axis $y \in [0, \infty)$ and unbounded on the whole half-axis; $u(0) = 1$.

In addition, let $u(y)$ form only a finite number of waveguides (for the exact definition see in ([1]) or ([2])). For definiteness we shall assume that the first waveguide does not begin immediately at the surface. Figure 2 shows $u(y)$ with two waveguides (j_1 and j_2).

3. Given $X(p)$ and $T(p)$, (1) may be regarded as a system of equations for $u(y)$. We already know (see ([1]) or ([2])) that this system does not have a unique solution. But it may also have no solutions at all. The question arises: what restrictions must be imposed on the functions $X(p)$ and $T(p)$ so that some $u(y)$, satisfying the conditions II.2, would be a solution of system (1).

4. The answer is

Theorem 1. *In order that the curve $\Gamma\{2X(p), 2T(p)\}$, $p \in (0, 1)$, be a travel-time curve from a surface source for a velocity section $u(y)$, satisfying the conditions II.2, it is necessary and sufficient that the following conditions be fulfilled:*

A. *The functions $X(p)$ and $T(p)$: 1) are positive; 2) are differentiable almost everywhere; 3) $T'(p) - pX'(p) = 0$ almost everywhere on $(0,1)$; 4) for all p where $T(p)$ and $X(p)$ are nondifferentiable (except, possibly, for a finite number of them),*

$$X(p \pm 0) = X(p) = T(p \pm 0) = T(p) = \infty.$$

B. *The function $\tau(p) = T(p) - pX(p)$: 1) decreases monotonically; 2) $\tau(1-0) = 0$; 3) is continuous everywhere, except at the points p_i, $p_1 > p_2 > \dots > p_n$, where it has jumps*

$$\sigma_i = \tau(p_i - 0) - \tau(p_i + 0).$$

C. *The function*

$$\Phi(q) = \frac{2}{\pi} \int_q^1 \frac{X(p)\,dp}{\sqrt{p^2 - q^2}} :$$

1) is finite for all $q \in (0,1)$; 2) does not increase; 3) $\Phi(+0) = +\infty$; 4) there exists such a $C > 0$ that everywhere on (p_{k+1}, p_k), where $\Phi'(q)$ is finite, the inequality

$$\Phi'(q) < -Cq/\sqrt{p_k^2 - q^2}, \qquad 1 \leqslant k \leqslant n, \quad p_{n+1} = 0;$$

is satisfied; 5) the function $g(y)$, inverse to $\Phi(q)$, is piecewise twice continuously differentiable.

D. *The function*

$$\tau(p) + \int_p^1 \sqrt{z^2 - p^2}\, d\Phi(z)$$

is continuously differentiable for $p \neq p_i$, $i = 1, 2, \dots, n$.

5. Let us note some special features of the conditions II.4.

The function $\tau(p) = T(p) - pX(p)$ is given only on the set where $X(p)$ and $T(p)$ are finite. But since this set is dense in $(0,1)$ and $\tau(p)$ is continuous everywhere, except for a finite number of points where it has jumps, $\tau(p)$ is defined everywhere on $(0,1)$.

The number of discontinuities of $\tau(p)$ is equal to the number of waveguides.

If a waveguide is located at the surface, then $p_1 = 1$. In this case condition B.2) becomes B.2′) $\tau(1-0) = \sigma_1 > 0$.

Fig. 3

Figure 3: Fig. 3

Fig. 4

Figure 4: Fig. 4

6. Instead of the condition D, which is difficult to verify, let us introduce the condition

D′. $X(p) = \infty$ for no more than a countable set of values $p \in (0,1)$. This condition is not necessary; however, the totality of conditions A, B, C, and D′ is sufficient for Γ to be a hodograph.

7. From condition C the following consequence follows:

$$\overline{\lim_{p\to p^0-0}} X(p) \geq \underline{\lim_{p\to p^0+0}} X(p) \quad \text{for any } p^0 \in (0,1).$$

Therefore the curve in Fig. 3 is not a hodograph.

III.1. Let us pass to the hodograph from a deep source. Applying transformation I.3, we may consider the problem immediately in the half-plane. Put

$$f(y) = \left(\sup\{u(y^0),\, 0 \leq y^0 \leq y\}\right)^{-1}.$$

Let the depth of the source be $y = d$, and let $f(d-0) = P,\ f(d+0) = Q$; it is clear that $P \geq Q$.

Fig. 3

Fig. 4

The rays going to the right upward from the source give a part of the hodograph $\Gamma_1\{X_1(p), T_1(p)\}$, where

$$X_1(p) = \int_0^d \frac{p\,u(y)\,dy}{\sqrt{1-p^2u^2(y)}},$$

$$T_1(p) = \int_0^d \frac{dy}{u(y)\sqrt{1-p^2u^2(y)}},$$

$$p \in [0, P].$$

If $X_1(Q) < \infty$, then Γ_1 (see Fig. 4) is the arc OI (when $Q = P$) or OJ (when $Q < P$).

2. Introduce the function $H(r) = \text{mes}\{y,\ y \le d,\ u(y) \le r\}$; then

$$X_1(p) = \int_0^{P^{-1}} \frac{pr\, dH(r)}{\sqrt{1-p^2r^2}}, \qquad T_1(p) = \int_0^{P^{-1}} \frac{dH(r)}{r\sqrt{1-p^2r^2}}, \qquad p \in [0, P]. \quad (2)$$

Equations (2) may be regarded as a system of equations with respect to $H(r)$. Obviously, $H(r)$ is a nondecreasing function and $H(0) = 0$.

3. The uniqueness of the solution of (2) is proved in ([2]). But the question arises: what properties must the functions $X_1(p)$ and $T_1(p)$ have in order that a solution exist in the class of nondecreasing functions. It is easy to see that this question is equivalent to the question of what conditions must be imposed on the curve Γ_1 so that it be part of a hodograph from a deep source. However, the restrictions on the velocity section $u(y)$ in this case are considerably weaker than in II.2. It is assumed that $u(y)$, $y \in (0, d)$, is positive, bounded, and measurable.

4. Introduce the quantities

$$\beta_i = \int_0^1 v_1^{iT}(vP)\, dv, \qquad i = 1, 2, \dots .$$

Define b_i, $i = 1, 2, ...$, from the triangular system of equations

$$\beta_{2k+1} = \frac{k!}{(2k+1)!} \sum_{i=1}^{k+1} \frac{(2k-i+1)!}{(k-i+1)!}\, b_i, \qquad k = 0, 1, 2, \dots .$$

Put $b_0 = T_1(0)/2$.

Lemma. $T_1(p)$ is representable in the form (2) with a nondecreasing function $H(r)$ if and only if the numbers b_i $(i = 0, 1, 2, ...)$ are a sequence of moments for the function

$$\mathcal{H}_1(z) = \int_1^z \frac{Pt}{4\sqrt{t-1}}\, dH\left(\frac{2\sqrt{t-1}}{Pt}\right), \qquad 1 < z < 2,$$

i.e.,

$$b_i = \int_1^2 z^i\, d\mathcal{H}_1(z). \quad (3)$$

Theorem 2. In order that the curve $\Gamma_1\{X_1(p), T_1(p)\}$, $p \in [0, P)$, be part of the traveltime curve from a deep source, it is necessary and sufficient that:

A. For every m the quadratic forms

$$\sum_0^m b_{i+j} x_i x_j, \qquad \sum_0^m (3b_{i+j+1} - 2b_{i+j} - b_{i+j+2}) x_i x_j$$

be nonnegative.

B. The functions $X_1(p)$, $T_1(p)$ be differentiable for $p \in [0, P)$.

C. $T_1'(p) - pX_1'(p) = 0$.

D. $X_1(0) = 0$.

Condition A is equivalent to condition A′ of nonnegativity of the forms

$$\sum_0^m (b_{i+j+1} - b_{j+i}) x_i x_j, \qquad \sum_0^m (2b_{i+j} - b_{i+j+1}) x_i x^j$$

and expresses the fact (see (3)) that b_i is a sequence of moments[3].

5. Theorems 1 and 2 also give us necessary and sufficient conditions in order that: a) a curve in the x, t plane be the traveltime curve of a pulse reflected from a boundary; b) a part Γ_2 of the traveltime curve from a deep source (see Fig. 4) correspond to some section $u(y)$.

Indeed, the traveltime curve of reflected waves is a curve Γ_1, stretched by a factor of 2 along both axes, for a source situated at the same depth as the reflecting boundary (see (2)).

As for the curve $\Gamma_2\{X_2(p), T_2(p)\}$, $p \in (0, Q)$, then, as shown in (2), the functions $X_d(p) = [X_2(p) - X_1(p)]/2$ and $T_d(p) = [T_2(p) - T_1(p)]/2$ are analogous to $X(p)$ and $T(p)$, if the surface is transferred to the depth $y = d$. Consequently, they must satisfy the conditions II.4 with changes caused by the fact that $p \in (0, Q)$, and not $(0, 1)$.

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Received
15 VII 1966

CITED LITERATURE

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Note: Figure translations are in progress. See original paper for figures.