ON THE GROWTH OF A FUNCTION \(\Phi(r_1,\ldots,r_n)\), CONVEX WITH RESPECT TO \(\ln r_1,\ldots,\ln r_n\)

MATHEMATICS

1967

SovietRxiv

Abstract

Full Text

UDC 517.55

MATHEMATICS

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ON THE GROWTH OF A FUNCTION $\Phi(r_1, \dots, r_n)$, CONVEX WITH RESPECT TO $\ln r_1, \dots, \ln r_n$

(Presented by Academician Yu. V. Linnik on 11 IV 1966)

As is known, there are various approaches to the study of the growth of entire functions of several variables. One of them is the study of the growth of an entire function $f(z_1, \dots, z_n)$ by considering the function

$$M_f(r_1, \dots, r_n) = \max_{|z_i|=r_i,\ i=1,\dots,n} |f(z_1, \dots, z_n)|.$$

Various characteristics of the growth of the function $f(z_1, \dots, z_n)$, constructed from the function $M_f(r_1, \dots, r_n)$, were introduced and investigated by W. Valiron (^{1}), M. M. Dzhrbashyan (^{2}), A. A. Gol' dberg (^{3}), L. S. Maergoiz (^{4}), the author ($^{5-7}$), and others (see also (^{8})). In this connection, Valiron' s results were based on the convexity, obtained by him, of the function $\ln M_f(r_1, \dots, r_n)$ with respect to $\ln r_1, \dots, \ln r_n$, while the results of the other authors were obtained with the aid of an easily established relation between the growth of the function $\ln M_f(r_1, \dots, r_n)$ and the behavior of the coefficients in the power-series expansion of the function $f(z_1, \dots, z_n)$. Let us note that in the theory of entire functions there naturally arises the need to study the growth of certain other special functions, for example the function

$$N_f(r_1, \dots, r_n) = \frac{1}{(2\pi)^n} \int_0^{2\pi} \cdots \int_0^{2\pi} \ln \left| f\left(r_1 e^{i\varphi_1}, \dots, r_n e^{i\varphi_n}\right) \right| d\varphi_1 \cdots d\varphi_n,$$

which characterizes the distribution of the zeros of the entire function $f(z_1, \dots, z_n)$*. It is clear that the study of the growth of the function $N_f(r_1, \dots, r_n)$ cannot be based on the use of the power-series expansion of the function $f(z_1, \dots, z_n)$. At the same time, as is not difficult to see, the function $N_f(r_1, \dots, r_n)$ is plurisubharmonic and, consequently (see, for example, (^{10})), convex with respect to $\ln r_1, \dots, \ln r_n$. It turns out that this is sufficient to extend to the function under consideration, and also to all functions possessing the indicated convexity, not only Valiron' s results, but also the results obtained in ([3],[4],[7]). Moreover, as will be seen below, results are obtained which are new also for the function $M_f(r_1, \dots, r_n)$.

Let $\mathfrak{M}$ be the set of functions $\Phi(r_1, \dots, r_n)$, defined for $r_1 > 0, \dots, r_n > 0$, convex with respect to $\ln r_1, \dots, \ln r_n$ and monotonically increasing in each of the variables $r_1, \dots, r_n$**. We introduce the following growth characteristics of such functions.

1. By the **order of the function** $\Phi(r_1, \dots, r_n) \in \mathfrak{M}$ **with respect to the totality of variables** we shall mean the number ρ_Φ defined by the equality

$$\rho_\Phi = \overline{\lim_{r \to \infty}} \frac{\ln \Phi(r, \dots, r)}{\ln r}.$$

* Some assertions on the growth of the function $N_f(r_1, \dots, r_n)$ are given in ([9]).

** Otherwise, the set $\mathfrak{M}$ may be characterized as the set of such functions $\Phi(r_1, \dots, r_n)$ that the functions $\Phi(|z_1|, |z_2|, \dots, |z_n|)$ are plurisubharmonic.

2. The **order of a function** $\Phi(r_1, \dots, r_n) \in \mathfrak{M}$ with respect to the variable r_i is the number $\tilde{\rho}_i(\Phi) \leqslant \infty$ defined by the equality

$$\tilde{\rho}_i(\Phi) = \sup_{r_1, \dots, r_{i-1}, r_{i+1}, \dots, r_n} \rho_{i,\Phi}(r_1, \dots, r_{i-1}, r_{i+1}, \dots, r_n),$$

$$\rho_{i,\Phi}(r_1, \dots, r_{i-1}, r_{i+1}, \dots, r_n) = \overline{\lim_{r_i \to \infty}} \frac{\ln \Phi(r_1, \dots, r_n)}{\ln r_i}.$$

The **type of a function** $\Phi(r_1, \dots, r_n)$ for order $\tilde{\rho}_i < \infty$ with respect to the variable r_i is the number $\tilde{\sigma}_i(\Phi) \leqslant \infty$ defined by the equality

$$\tilde{\sigma}_i(\Phi) = \sup_{r_1, \dots, r_{i-1}, r_{i+1}, \dots, r_n} \sigma_{i,\Phi}(r_1, \dots, r_{i-1}, r_{i+1}, \dots, r_n),$$

where

$$\sigma_{i,\Phi}(r_1, \dots, r_{i-1}, r_{i+1}, \dots, r_n) = \overline{\lim_{r_i \to \infty}} \frac{1}{r_i^{\rho_i}} \Phi(r_1, \dots, r_n).$$

3. The **hypersurface of conjugate orders** of the function $\Phi(r_1, \dots, r_n) \in \mathfrak{M}$ is the boundary S_ρ of the set $B_\rho \subset R^n$ formed by the points $(a_1, \dots, a_n)$ for which, for some $C = C(a_1, \dots, a_n)$ and all $r_1 > 0, \dots, r_n > 0$, the inequality

$$\Phi(r_1, \dots, r_n) \leqslant C + r_1^{a_1} + \dots + r_n^{a_n}$$

holds.

Numbers $\rho_1, \dots, \rho_n$ such that $(\rho_1, \dots, \rho_n) \in S_\rho$ will be called **conjugate orders** of the function $\Phi(r_1, \dots, r_n)$.

The **hypersurface of conjugate types for conjugate orders** $\rho_1, \dots, \rho_n$ is the boundary S_σ of the set $B_\sigma \subset R^n$, formed by the points $(a_1, \dots, a_n)$ for which, for some $C = C(a_1, \dots, a_n)$ and all $r_1 > 0, \dots, r_n > 0$, the inequality

$$\Phi(r_1, \dots, r_n) \leqslant C + a_1 r_1^{\rho_1} + \dots + a_{nr} n^{\rho_n}$$

is satisfied.

Numbers $\sigma_1, \dots, \sigma_n$ such that $(\sigma_1, \dots, \sigma_n) \in S_\sigma$ will be called a **system of conjugate types for the conjugate orders** $\rho_1, \dots, \rho_n$.

Theorems 1 and 2, formulated below, were essentially already contained in (^{1}).

Theorem 1. For any function $\Phi(r_1, \dots, r_n) \in \mathfrak{M}$ the inequality

$$\rho_\Phi \leqslant \tilde{\rho}_1(\Phi) + \dots + \tilde{\rho}_n(\Phi)$$

holds.

Theorem 2. For any function $\Phi(r_1, \dots, r_n) \in \mathfrak{M}$ the identity

$$\rho_{i,\Phi}(r_1, \dots, r_{i-1}, r_{i+1}, \dots, r_n) \equiv \tilde{\rho}_i(\Phi)$$

holds.

The following theorems 3, 4, and 5 are analogous to the corresponding theorems for entire functions ($^{5-7}$), differing from them in proof.

Theorem 3. Let $\Phi(r_1, \dots, r_n) \in \mathfrak{M}$ and $\tilde{\sigma}_i(\Phi) < \infty$. Then for all $r_1 > 0, \dots, r_n > 0$

$$\sigma_{i,\Phi}(r_1, \dots, r_{i-1}, r_{i+1}, \dots, r_n) = \tilde{\sigma}_i(\Phi).$$

Theorem 4. Let S be some hypersurface lying in the hyperoctant $\{a_i \geqslant 0,\ i = 1, 2, \dots, n\}$. Further, let S^{-1} be the hypersurface obtained from S by the transformation $b_i = 1/a_i,\ i = 1, \dots, n$. Then

for the hypersurface S to be a hypersurface of conjugate orders for at least one function $\Phi(r_1, \dots, r_n) \in \mathfrak{M}$, it is necessary and sufficient that S^{-1}, together with the coordinate planes, form the boundary of some convex domain containing, together with each of its points $(b'_1, \dots, b'_n)$, all points $(b_1, \dots, b_n)$ for which

$$0 < b_i \leq b'_i, \qquad i = 1, \dots, n.$$

Theorem 5. Let S be a hypersurface situated in the hyperoctant $\{a_i \geq 0,\ i = 1, \dots, n\}$. Let, further, S_1 be the hypersurface obtained from S by the transformation $b_i = \ln a_i,\ i = 1, 2, \dots, n$. Then, in order that the hypersurface S be a hypersurface of conjugate types for at least one function $\Phi(r_1, \dots, r_n) \in \mathfrak{M}$, it is necessary and sufficient that the hypersurface S_1 be the boundary of some convex domain containing, together with each of its points $(a'_1, \dots, a'_n)$, the entire hyperoctant $\{a_i \geq a'_i,\ i = 1, \dots, n\}$.

The proofs of these theorems are close and are based on the use of the inequality

$$\Phi(t_1^\lambda s_1^\mu, \dots, t_n^\lambda s_n^\mu) \leq \lambda \Phi(t_1, \dots, t_n) + \mu \Phi(s_1, \dots, s_n) \tag{1}$$

where $\lambda \geq 0,\ \mu \geq 0,\ \lambda + \mu = 1$, whose validity is ensured by the membership of the function $\Phi(r_1, \dots, r_n)$ in the class $\mathfrak{M}$. We shall give here only the proof of Theorem 5.

First of all, let us note that sufficiency follows from the corresponding theorem on entire functions ([7],[8]), in which it is shown that for a hypersurface S having the properties indicated in Theorem 5 there exists an entire function whose hypersurface of conjugate types is S. To prove necessity, put

$$t_i = r_i \exp\left[\frac{\mu}{\rho_i}(\ln b_i - \ln a_i)\right], \qquad s_i = r_i \exp\left[\frac{\lambda}{\rho_i}(\ln a_i - \ln b_i)\right], \qquad i = 1, \dots, n,$$

where $\lambda \geq 0,\ \mu \geq 0,\ \lambda + \mu = 1,\ (a_1, \dots, a_n) \in B_\sigma$ and $(b_1, \dots, b_n) \in B_\sigma$. Then, by inequality (1) and the definition of the set B_σ, we have

$$\Phi(r_1, \dots, r_n) = \Phi(t_1^\lambda s_1^\mu, \dots, t_n^\lambda s_n^\mu) \leq \lambda \Phi(t_1, \dots, t_n) + \mu \Phi(s_1, \dots, s_n) \leq$$

$$\leq \lambda \left(C_1 + \sum_{i=1}^n a_i t_i^{\rho_i} \right) + \mu \left(C_2 + \sum_{i=1}^n b_i s_i^{\rho_i} \right) =$$

$$= C_3 + \lambda \sum_{i=1}^n a_i^\lambda b_i^\mu r_i^{\rho_i} + \mu \sum_{i=1}^n a_i^\lambda b_i^\mu r_i^{\rho_i} = C_3 + \sum_{i=1}^n a_i^\lambda b_i^\mu r_i^{\rho_i}.$$

It follows that for any $\lambda \in [0, 1]$ the point $(a_1^\lambda b_1^\mu, \dots, a_n^\lambda b_n^\mu) \in B_\sigma$. The theorem is proved.

For functions $\Phi(r_1, \dots, r_n) \in \mathfrak{M}$ having finite order in the aggregate of variables, the following theorems have been obtained:

Theorem 6. Let the function $\Phi(r_1, \dots, r_n) \in \mathfrak{M}$ have order $\rho < \infty$ in the aggregate of variables, and, with respect to the variable r_n, order $\rho_n < \infty$. Then, if for some $r_1^0 > 0, \dots, r_{n-1}^0 > 0$

$$\sigma_{n,\Phi}(r_1^0, \dots, r_{n-1}^0) \le \gamma,$$

then there exists a constant $c = c(r_1^0, \dots, r_{n-1}^0)$ such that for all $r > 0$ the inequality

$$\sigma_{n,\Phi}(r, \dots, r) \le \gamma c r^{\rho - \tilde{\rho}_n}$$

holds.

Theorem 7. Let the function $\Phi(r_1, \dots, r_n) \in \mathfrak{M}$ have order $\rho < \infty$ in the aggregate of variables, and, with respect to the variable r_n, order $\tilde{\rho}_n < \infty$.

Then, if the integral

$$\int^{\infty} \frac{\Phi(r_1, \dots, r_n)}{r_n^{\rho_n + 1}} \, dr_n$$

converges for some values $r_1 = r_1^0 > 0, \dots, r_{n-1} = r_{n-1}^0 > 0$, then it also converges for any $r_1 > 0, \dots, r_{n-1} > 0$.

We note that Theorem 6, for the case of entire functions, gives an estimate of the type $\sigma_{n,\Phi}(r_1, \dots, r_n)$ more precise than the one following from the result recently obtained by us (^{11}). Theorem 7 has no analogue among the theorems on entire functions known at the present time. We omit the proofs of these theorems. We indicate only that the estimates needed for the proof, as in the preceding cases, are based on inequality (1). In doing so we put

$$t_i = r^{\alpha_i/(\lambda\alpha_i + \mu\beta_i)}, \quad s_i = r^{\beta_i/(\lambda\alpha_i + \mu\beta_i)}, \quad i = 1, \dots, n-1;$$

$$t_n = r_n^{\alpha_n/(\lambda\alpha_n + \mu\beta_n)}, \quad s_n = r_n^{\beta_n/(\lambda\alpha_n + \mu\beta_n)},$$

where $\alpha_n = \tilde{\rho}_n - \varepsilon,\ \beta_n = \rho,\ \dfrac{\beta_i}{\alpha_i} = \dfrac{1}{\ln r_n - c(r)},\ c(r) = \dfrac{\rho}{\tilde{\rho}_n - \varepsilon} \ln r - 1,\ \lambda = \dfrac{c(r)}{\ln r_n},$

$$\mu = 1 - \lambda.$$

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Received
9 IV 1966

REFERENCES

[1] G. Valiron, Bull. Sci. Math., **47**, 177 (1962).
[2] M. M. Dzhrbashyan, Izv. AN ArmSSR, ser. phys.-math., **8**, No. 4, 1 (1955).
[3] A. A. Gold' berg, Dokl. AN ArmSSR, **29**, 145 (1959).
[4] L. S. Maiergoiz, Uch. zap. Moscow Regional Pedagogical Institute, **137**, 8, 133 (1964).
[5] L. I. Ronkin, Matem. sborn., **39**, 2, 253 (1956).
[6] L. I. Ronkin, Zap. matem. otd. fiz.-matem. fak. Kharkov. univ. i Kharkovsk. matem. obshch., **27**, ser. 4, 59 (1961).
[7] L. I. Ronkin, Ukr. matem. zhurn., **3**, 408 (1964).
[8] B. A. Fuks, *Introduction to the Theory of Analytic Functions of Several Complex Variables*, Moscow, 1961.
[9] P. Lelong, Ann. Éc. norm. sup., **58**, 83 (1941).
[10] V. S. Vladimirov, *Methods of the Theory of Functions of Several Complex Variables*, "Nauka," 1964.
[11] L. I. Ronkin, DAN, **169**, No. 3, 529 (1966).

Note: Figure translations are in progress. See original paper for figures.

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