Bounded solutions of second-order differential equations

A. I. Perov

1967

SovietRxiv

Abstract

Full Text

Preamble

DIFFERENTIAL EQUATIONS 1967, VOLUME III, No. 3 BOUNDED SOLUTIONS OF SECOND-ORDER DIFFERENTIAL EQUATIONS A. I. PEROV

Let $\mathfrak{B}$ be the Banach space of all complex-valued functions that are continuous and bounded on the real line, endowed with the norm $\|f\| = \sup_t |f(t)|$. It can be shown (see, for example, [53–57]) that the differential equation

$$Lx(t) = \ddot{x} + a_1\dot{x} + a_2 x = f(t) \tag{1}$$

possesses a unique bounded solution for any $f \in \mathfrak{B}$ if and only if the characteristic roots of the operator lie off the imaginary axis. In this case, the bounded solution and its derivative are given by the following formulas:

$$x(t) = \int_{-\infty}^{\infty} G(t-s)f(s)ds, \tag{2}$$

$$\dot{x}(t) = \int_{-\infty}^{\infty} \dot{G}(t-s)f(s)ds, \tag{3}$$

where the Green's function depends only on the operator. The integral operators defined above act in the space denoted by $G(Q)$ and $\dot{G}(Q)$, respectively. In the following discussion, it is assumed that the characteristic numbers are real and distinct. We distinguish between the **real case**, where the characteristic numbers are real, and the **complex case**, where the characteristic numbers are complex conjugates, denoted as $\lambda = \alpha \pm i\beta$.

The primary result of this article is the following assertion:

Theorem

Suppose the characteristic numbers of the operator lie on the imaginary axis. Then the following relations hold:

1. The Real Case

In the case where the characteristic numbers are real, the following relations are valid:

$\|G(L)\| =$

$|\lambda_i|^{-1} - |\lambda|$ (complex case) or $|\lambda_i| - |\lambda|$ (real case),

$\|P(L)\| = \sum 2/e|\lambda_i|$,

arctan (complex case). In formula (5), for the real case, the expression for the norm of the integral operator is provided first for characteristic numbers of the same sign, then for characteristic numbers of different signs, and finally for a multiple characteristic number.

Before proceeding to the proof of the theorem, we make a general remark regarding integral operators with kernels depending on the difference of their arguments. Let $T(t)$ be a real measurable function that is integrable on $(-\infty, +\infty)$. Then the integral operator K defined by the formula

$$Kf(t) = \int_0^{+\infty} T(t-s)f(s)ds, \tag{6}$$

acts in a Banach space and is a linear bounded operator, where

$$\|K\| = \int_{-\infty}^{\infty} |T(a)|da. \tag{7}$$

Indeed, from the obvious inequalities

$$|f(x)| \le \|f\|,$$

$$|E(f) - E_n(f)| \le B\|f\|/n,$$

it follows that the function is bounded and uniformly continuous. Furthermore, $\|Kf\| \le C_k\|f\|$ (where we denote the right-hand side of relation (7) by C_k). Suppose an arbitrary $\epsilon > 0$. Let us choose a number R such that

$$\int_{|a|>R} |T(a)|da < \frac{\epsilon}{3}, \tag{8}$$

By virtue of the absolute continuity of the integral, we can find a number $\delta > 0$ such that for any measurable set e with measure $\mu(e) < \delta$, the inequality $\int_e |T(a)|da < \frac{\epsilon}{3}$ holds. Subsequently, applying Luzin' s theorem [**?**], we construct a continuous real-valued function $\psi(a)$ possessing the following properties: the measure of the set where the function deviates from the sign of the original function, specifically $\mu\{a : |a| < R, \psi(a) \neq \text{sgn } T(a)\}$, is less than δ. From this, we obtain:

$$|\int T(a)\psi(-a)da| > C_k - \epsilon$$

and, according to formulas (8), (9), and (10), we have:

$$\frac{4}{\pi}\int_0^\infty \left|\frac{2}{\pi}\Gamma(a)\right| da - q\Gamma(a)\left|(1-a)\Gamma(a)\right|$$

$$\int |T(a)\psi(-a) - |T(a)||da < \epsilon$$

Thus, $|\Delta| > 0$ because $\|\phi\| > (k-\epsilon)\|\phi\|$. Combined with the previously obtained inequality, this confirms the validity of the relation. Our assertion is proven. We now proceed to the proof of the theorem. First, let us consider the real case. Suppose the characteristic numbers are real, distinct, and of the same sign. For the sake of definiteness, assume $0 < \lambda_1 < \lambda_2$. Then the formula takes the form $\int_{-\infty}^{t} f(s)ds$, and for the Green' s function, we obtain the expression:

$$G(a) = \begin{cases} (\lambda_1 - \lambda_2)^{-1}(e^{\lambda_1 a} - e^{\lambda_2 a}) & \text{for } a < 0, \\ 0 & \text{for } a > 0. \end{cases} \tag{12}$$

Suppose the characteristic numbers are real, distinct, and of opposite signs. Let us assume $\lambda_2 < 0 < \lambda_1$. In this case, the formula takes the form $\int_{-\infty}^{+\infty} f(s)ds$, and consequently, $G(a)$ is defined for $a < 0$ and $a > 0$. Now, suppose the characteristic numbers coincide, i.e., $\lambda_1 = \lambda_2 = \lambda$. For definiteness, assume $\lambda > 0$. The formula then becomes $\int_{-\infty}^{+\infty} tf(s)ds$, and we obtain the following expression for the Green' s function: $G(a)$ for $a < 0$ and $G(a)$ for $a > 0$. According to the results in Section 2, the integrals of $|G(a)|$ are calculated in an elementary manner, yielding the expressions for the norms of the integral operators $J(f)$. Finally, let us consider the complex case. Suppose the characteristic numbers lie in the right half-plane, where $\lambda_{1,2} = \alpha \pm i\beta$ with $\alpha > 0$. The formula then takes the form:

$$x(t) = \int_{-\infty}^{t} G(t-s)f(s)ds \tag{17}$$

and, consequently, $G(a)$ is defined for $a > 0$.

According to our rule for determining the norm of an integral operator, we obtain $\|G(\xi)\|$. The desired expression for $\|G(\xi)\|$ is derived through the following sequence of calculations:

$$\|G(\xi)\| = \sqrt{2\int_0^\infty |e^{-a\alpha}\sin\beta\alpha|^2 d\alpha}$$

1 Introduction

In recent years, the rapid advancement of machine learning and deep learning has revolutionized various scientific and engineering disciplines. These computational frameworks have moved beyond simple data processing to become essential tools for modeling complex physical systems, optimizing industrial processes, and uncovering hidden patterns in high-dimensional datasets. As the complexity of these models increases, the demand for robust, scalable, and interpretable algorithms has never been higher.

The integration of domain-specific knowledge into neural network architectures represents a significant shift in the field. Rather than treating models as "black boxes," researchers are increasingly incorporating physical constraints, statistical priors, and structural symmetries to improve generalization and reliability. This approach is particularly critical in fields where data may be sparse, noisy, or expensive to collect, as it allows the model to leverage existing theoretical foundations to guide the learning process.

[Figure 1: Neural network architecture with physical constraints]

Furthermore, the scalability of these algorithms remains a primary concern for practical applications. While deep learning models have demonstrated remarkable performance on benchmark datasets, their deployment in real-time systems requires careful consideration of computational efficiency and memory constraints. Recent developments in hardware acceleration and algorithmic optimization have begun to bridge this gap, enabling the application of sophisticated machine learning techniques to large-scale, real-world problems.

[Table 1: Performance comparison of different architectures]

In this paper, we explore the intersection of these methodologies, focusing on the development of novel architectures that balance flexibility with theoretical rigor. We provide a comprehensive analysis of current trends, identify key challenges in the field, and propose potential directions for future research. By synthesizing insights from both classical statistics and modern deep learning, we aim to provide a framework that enhances both the predictive power and the interpretability of scientific models.

$$\int e^{-\alpha s}(\sin\beta s - \beta\cos\beta s)ds$$

Furthermore, since $G(\alpha) = \frac{1}{\beta}e^{\alpha\alpha}\sin(\beta\alpha + \delta\cos\beta\alpha)$ for $\alpha < 0$, we proceed with the calculation of this integral as follows: we set $\phi = \arctan\frac{\beta}{\alpha}$, where $0 < \phi < \frac{\pi}{2}$, and perform a change of variables using the formula $\tau = \alpha + \frac{\phi}{\beta}$. We then obtain:

$$\|G(\beta)\| = \frac{1}{\beta}\int e^{\alpha\tau}|\sin\beta\tau|d\tau$$

and, consequently,

$\|\mathcal{A}\Phi_0 I_6(\xi)\| = e^{P/\sqrt{2}+P^2/\sqrt{e}}$

$|\sin pT|dT|\sin ... \arctan\{\frac{TF(f)}{Tf(t)}\}$. The case where the characteristic numbers lie in the left half-plane is treated analogously. The theorem is proved. We now consider a nonlinear second-order equation:

$$\ddot{x}(t) = \varphi(t, x, \dot{x}) \tag{22}$$

where $\varphi(t, x, y)$ is defined for $|x|, |y| < +\infty$, is continuous in t, and satisfies the Lipschitz condition with respect to the variables x and y: $|\varphi(t, x_1, y_1) - \varphi(t, x_2, y_2)| \leq L_1|x_1 - x_2| + L_2|y_1 - y_2|$. Furthermore, it is assumed that $\varphi(t, 0, 0) \equiv 0$.

Theorem (Perov). Suppose that under the assumptions made above, the characteristic numbers of the operator lie on the imaginary axis and the inequality $\|L_1\| \cdot \|u(t)\| \leq ...$ holds. Then the equation has a unique bounded solution. For this solution, the following estimates are valid:

$$\|x\| \leq C\|\varphi(t, 0, 0)\| \tag{25}$$

The solution to the equation can be obtained using the method of successive approximations:

$$x_{n+1}(t) = \int_0^t G(t - s)\phi(s, x_n(s), \dot{x}_n(s))ds$$

for $n = 0, 1, 2, ...$, starting from an arbitrary element (t, f). The proof is based on reducing the problem of finding a bounded solution to an integral equation, to which a generalized contraction mapping principle is then applied.

We make the following observation regarding this theorem: if $\phi(t, x, y)$ possesses the property of ω-periodicity, then the bounded solution is also ω-periodic; if $\phi(t, x, y)$ is almost-periodic in t, then the bounded solution is also almost-periodic. This is explained by the fact that the integral operators encountered leave the subspaces of ω-periodic functions and almost-periodic functions invariant (a technique we have utilized previously [**?**]).

Equation (22) for $\phi(t, x, y) = f(t) + a(t)x + b(t)y$ was studied in work [**?**]. In that study, it was assumed that the characteristic numbers of the operator are distinct ($\lambda_i \neq \lambda_j$ for $i, j = 1, 2$); the condition for the existence of a unique bounded solution was written as:

$$(|\lambda_1| + |\lambda_2|)^{-1} \cdot |a - b| \cdot \|f\| < 1$$

It is easy to see that this condition is more restrictive than the inequality (24) obtained in our work.

References

1. Krein, M. G. *Lectures on the Theory of Stability of Solutions of Differential Equations in Banach Space.* Kiev.
2. Natanson, I. P. *Theory of Functions of a Real Variable.* GITTL.
3. Perov, A. I. “On the Cauchy Problem for a System of Ordinary Differential Equations.” *Approximate Methods for Solving Differential Equations*, Kiev, “Naukova Dumka” .
4. Perov, A. I. *Doklady Akademii Nauk SSSR (DAN SSSR)*, 1960.

Gelman

Leningrad Institute of Electrical Engineering, Vol. 35, 1958. Received by the editorial office of Voronezh State University on April 5.

Note: Figure translations are in progress. See original paper for figures.