# ON TRIANGULAR REPRESENTATIONS OF LINEAR OPERATORS AND MULTIPLICATIVE REPRESENTATIONS OF THEIR CHARACTERISTIC FUNCTIONS

MATHEMATICS

1967

SovietRxiv

## Abstract

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UDC 517.432

**MATHEMATICS**

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# ON TRIANGULAR REPRESENTATIONS OF LINEAR OPERATORS AND MULTIPLICATIVE REPRESENTATIONS OF THEIR CHARACTERISTIC FUNCTIONS

*(Presented by Academician S. L. Sobolev on 29 VIII 1966)*

In recent years the theory of triangular representations of operators and multiplicative representations of their characteristic functions has been developed ($^{1-6}$). In all these investigations it was required of the operator represented in triangular form that it differ from a self-adjoint or unitary operator by a completely continuous summand. It turns out that there exist important and broad classes of operators for which essentially all the constructions used in deriving triangular and multiplicative representations go through, although the indicated condition is not satisfied. As was shown in the authors' paper ($^{7}$), these classes include, in particular, simple dissipative operators similar to self-adjoint ones, and simple contractions similar to unitary operators.

1. Let $\mathfrak{R}$ denote the ring of all bounded linear operators acting in a separable Hilbert space $\mathfrak{H}$, and let $\mathfrak{S}_\infty$ denote its ideal consisting of all completely continuous operators.

Let $\mathfrak{P} = \{P\}$ be some chain of orthoprojections (see ($^{6}$)). If for an operator $A \in \mathfrak{R}$ the function

$$\Phi(A;\mathfrak{z}) = \sum_j \Delta P_j A \Delta P_j,$$

where $\mathfrak{z} = \{P_j\}_0^n$ is a partition of the chain $\mathfrak{P}$, and $\Delta P_j = P_j - P_{j-1}$ ($j = 1, 2, \ldots, n$), has a limit in the uniform topology (in the sense of Shatunovskii, see ($^{6}$), Chap. I), then this limit will be denoted by

$$\int_{\mathfrak{P}} dP\,A\,dP$$

and called the **uniform diagonal of the operator $A$ along the chain $\mathfrak{P}$.**

**Theorem 1.** *The following assertions are equivalent:*

1°. *The operator* $V(\in \mathfrak{R})$ *has a proper chain* $\mathfrak{P}$*, and its uniform diagonal along* $\mathfrak{P}$ *is equal to zero.*

2°. *The operator* $V$ *is quasinilpotent (i.e.,* $\sigma(V) = \{0\}$*), has a proper chain* $\mathfrak{P}$*, and the uniform diagonal of its imaginary component* $V_{\mathfrak{J}} = (V - V^*)/2i$ *is equal to zero.*

3°. *The operator* $V$ *admits a uniformly convergent triangular representation*

$$V = 2i \int_{\mathfrak{P}} P V_{\mathfrak{J}} \, dP = 2 \int_{\mathfrak{P}} P V_{\mathfrak{R}} \, dP. \tag{1}$$

We note that if some operator $V(\in \mathfrak{R})$ admits a uniformly convergent representation

$$V = 2i \int_{\mathfrak{P}} P H \, dP \left( = 2 \int_{\mathfrak{P}} P G \, dP \right), \tag{2}$$

where $H = H^* \in \mathfrak{R}$ $(G = G^* \in \mathfrak{R})$, then $V$ has property 1° and $H = V_{\mathfrak{J}}$ $(G = V_{\mathfrak{R}})$. Thus, every operator from $\mathfrak{R}$ with zero uniform

diagonal along the proper chain $\mathfrak{P}$ is uniquely recovered from its chain and one of its Hermitian components.

Let us make the following further remark. Let $V(\in \mathfrak{R})$ be an operator with proper chain $\mathfrak{P}$, along which its uniform diagonal is equal to zero. Then every chain $\mathfrak{P}' \supset \mathfrak{P}$ is proper for $V$, and along it the uniform diagonal of $V$ is equal to zero.

2. Let the monotone chain $\{P_t\}_{-\infty \leq t \leq \infty}$ $(P_{-\infty} = 0,\ P_{\infty} = I;\ P_{t_1} \leq P_{t_2}$ for $t_1 < t_2)$ be a proper chain of some operator $A(\in \mathfrak{R})$ with real spectrum. We shall say that the chain $\{P_t\}_{-\infty \leq t \leq \infty}$ **separates the spectrum** $\sigma(A)$ **of the operator** $A$, if the spectrum of the restriction of $A$ to the subspace $P_t \mathfrak{H}$ lies on the half-axis $(-\infty, t]$, and the spectrum of the restriction of the operator $(I - P_t) A (I - P_t)$ to the subspace $(I - P_t)\mathfrak{H}$ lies on the complementary half-axis $[t, \infty)$.

On the basis of one of the theorems of Yu. I. Lyubich and V. I. Macaev (8), one may assert, in particular, that every operator $A(\in \mathfrak{R})$ with real spectrum has a proper chain separating its spectrum, provided the condition

$$\int_0^{\infty} \ln^+ \ln^+ M(\delta) \, d\delta < \infty, \qquad \text{where } M(\delta) = \max_{|\lambda| = \delta} \|(A - \lambda I)^{-1}\|. \tag{3}$$

is satisfied.

As V. I. Macaev showed, condition (3) is satisfied every time when $A_{\mathfrak{J}} \in \mathfrak{S}_\omega$.

We shall assign an operator $A \in \mathfrak{R}$ with real spectrum to the class $\mathfrak{A}(\mathfrak{P})$ if it has the following properties: a) the chain $\mathfrak{P}$ is proper for $A$; b) from the projectors of the chain $\mathfrak{P}$ one can compose a chain separating the spectrum of the operator $A$; c) the uniform diagonal of the imaginary component $A_{\mathfrak{J}}$ of the operator $A$ along $\mathfrak{P}$ is equal to zero.

M. S. Brodskii (1) showed that if the chain $\mathfrak{P}$ is maximal (see (6)) and the imaginary component $A_{\mathfrak{J}} \in \mathfrak{P}_\infty$, then condition c) is a consequence of conditions a) and b).

**Theorem 2.** *In order that an operator $A(\in \mathfrak{R})$ be representable in the form $A = G + V$, where $V(\in \mathfrak{R})$ is an operator with proper chain $\mathfrak{P}$ and with uniform diagonal along it equal to zero, and $G$ is a self-adjoint operator defined by the equality*

$$G = \int_{\mathfrak{P}} a(P)\, dP,$$

*in which $a(P)$ $(P \in \mathfrak{P})$ is a nondecreasing left-continuous function, it is necessary and sufficient that the operator $A$ belong to the class $\mathfrak{A}(\mathfrak{P})$.*

Thus, every operator $A \in \mathfrak{A}(\mathfrak{P})$ is characterized by the fact that it admits the triangular representation

$$A = \int_{\mathfrak{P}} a(P)\, dP + 2i \int_{\mathfrak{P}} P A_{\mathfrak{J}}\, dP, \tag{4}$$

where $a(P)$ is the maximal point of the spectrum of the operator $A$ in the invariant subspace $P\mathfrak{H}$.

It follows from representation (4) that the class $\mathfrak{A}(\mathfrak{P})$ is a subalgebra of the algebra $\mathfrak{R}$.

3. Let the operator $A \in \mathfrak{R}$ and its imaginary component $A_{\mathfrak{J}}$ be represented in the form $A_{\mathfrak{J}} = RJR^*$, where $R$ is a linear bounded operator acting from some auxiliary Hilbert space $\mathfrak{H}_W$ into $\mathfrak{H}$, and $J$ is a signature operator acting in $\mathfrak{H}_W$, i.e. $J^* = J$ and $J^2 = I$. Following M. S. Brodskii (2), we form the operator-function

$$W(\lambda) = I + 2iJR^*(A^* - \lambda I)^{-1} R. \tag{5}$$

**Theorem 3.** *If the operator $A \in \mathfrak{A}(\mathfrak{P})$ and equality (4) gives its triangular representation, then its characteristic function (5) ad-*

admits the multiplicative representation

$$W(\lambda) = \int_{\mathfrak{P}} \left( I - 2iJ \, \frac{R^* \, dPR}{\lambda - a(P)} \right).$$

If the function $E(P) = R^*PR$ has bounded operator variation, i.e.

$$\int_{\mathfrak{P}} \|R^* \, dPR\| = \sup_{\mathfrak{z}} \sum_j \|R^* \Delta P_{jR}\| < \infty,$$

where $\mathfrak{z} = \{P_j\}_0^n$ is an arbitrary partition of the chain $\mathfrak{P}$, then the integral can be transformed to the form

$$W(\lambda) = \int_{\mathfrak{P}} \exp\left( -2iJ \, \frac{R^* \, dPR}{\lambda - a(P)} \right).$$

Theorems 1-3, under the assumption that the imaginary component $A_J \in \mathfrak{S}_\infty$, were proved by M. S. Brodskii ($^{1,2}$).

4. We now pass to the consideration of operators "close" to unitary ones. For any operator $T \in \mathfrak{R}$, let $D_T$ denote the deviation operator $I - T^*T$ of the operator $T$ from a unitary one. For lack of space, a theorem analogous to Theorem 1 is not given here.

For an operator $T (\in \mathfrak{R})$ with unitary spectrum (i.e., with spectrum on the unit circle), a chain separating the spectrum is defined analogously to the case of a real spectrum (see (3)).

By $\mathfrak{U}(\mathfrak{P})$ we denote the class of all operators $T \in \mathfrak{R}$ with unitary spectrum for which the following conditions are satisfied: $\alpha$) the chain $\mathfrak{P}$ is proper for $T$; $\beta$) from the projectors of $\mathfrak{P}$ one can form a chain separating the spectrum of the operator $T$; $\gamma$) the relation* holds

$$(M) \qquad \int_{\mathfrak{P}} dP \, D_T (I - PD_{TP})^{-1} \, dP = 0.$$

If the chain $\mathfrak{P}$ is maximal and $D_T \in \mathfrak{S}_\infty$, then condition $\gamma$) is a consequence of conditions $\alpha$) and $\beta$) (see (3)).

**Theorem 4.** *In order that an operator $T \in \mathfrak{R}$ be representable in the form $T = U(I + V)$, where $V (\in \mathfrak{R})$ is an operator with proper chain $\mathfrak{P}$ and with diagonal, uniform along it, equal to zero, while $U$ is the unitary operator defined by the equality*

$$U = \int_{\mathfrak{P}} \exp(i\varphi(P)) \, dP,$$

*in which* $\varphi(P)$ $(P \in \mathfrak{P};\ 0 \leq \varphi(P) \leq 2\pi)$ *is a nondecreasing left-continuous function, it is necessary and sufficient that the operator* $T \in \mathfrak{U}(\mathfrak{P})$*.*

5. Comparison of the operator $T \in \mathfrak{R}$ with a unitary one is carried out by means of its characteristic function $\theta_T(\zeta)$ (cf. $(^{10,11})$)

$$\theta_T(\zeta) = \left[T - \zeta J_{T^*} |D_{T^*}|^{1/2} (I - \zeta T^*) |D_T|^{1/2}\right] \big|_{\mathfrak{R}T},$$

where $J_T = \operatorname{sign} D_T$, $|D_T| = (D_T^2)^{1/2}$; the values of $\theta_T(\zeta)$ are regarded as operators acting from the subspace $\mathfrak{R}_T = \overline{D_T \mathfrak{H}}$ into the subspace $\mathfrak{R}_{T^*} = \overline{D_{T^*} \mathfrak{H}}$.

**Theorem 5.** *If the operator* $T \in \mathfrak{U}(\mathfrak{P})$*, then its characteristic function admits the multiplicative representation*

$$T^* \theta_T(\xi) = \int_{\mathfrak{P}} \left( I + J_{T^*} \frac{|D_T|^{1/2}\, dP\, (I - P D_{TP})^{-1} |D_T|^{1/2}}{\xi \exp(-i\varphi(P)) - 1} \right), \tag{6}$$

* This integral is understood as the limit of partial sums

$$\sum_j \Delta P_{jD} T (I - P_{jD_{TP}} j)^{-1} \Delta P_j$$

along the chain $\mathfrak{P}$.

where $\varphi(P)$ is a function, and $\mathfrak{P}$ is a chain from the triangular representation of the operator $T$.

If, in addition, the condition* is satisfied

$$\int_{\mathfrak{P}} \| \, |D_T|^{1/2}\, dP\, |D_T|^{1/2} \| < \infty, \tag{7}$$

then the integral (8) can be transformed to the form

$$T^* \theta_T(\xi) = \int_{\mathfrak{P}}^{\leftarrow} \exp \left( J_{T^*} \frac{|D_T|^{1/2}\, dP\, (I - P D_{TP})^{-1}\, |D_T|^{1/2}}{\xi \exp(-i\varphi(P)) - 1} \right). \tag{8}$$

By formal transformations, formulas (6) and (8) can be converted respectively to the form

$$\theta_T(\xi) = U \int_{\mathfrak{P}}^{\leftarrow} \left( I + \frac{1}{2} \frac{e^{i\varphi(P)} + \xi}{e^{i\varphi(P)} - \xi}\, J_{T^*} R^*(P)\, dP\, R(P) \right),$$

$$\theta_T(\xi) = U \int_{\mathfrak{P}}^{\leftarrow} \exp\left(\frac{1}{2}\frac{e^{i\varphi(P)}+\xi}{e^{i\varphi(P)}-\xi}\, J_{T^*} R^*(P)\, dP\, R(P)\right),$$

where $U$ is a unitary operator mapping $\mathfrak{R}_T$ onto $\mathfrak{R}_{T^*}$, and

$$R(P) = |D_T|^{1/2} \int_{\mathfrak{P}_P}^{\leftarrow} \left(I + \frac{1}{2}|D_T|^{1/2}(I - PD_{TP})^{-1}|D_T|^{1/2}\right);$$

by $\mathfrak{P}_P$ we denote the part of the chain $\mathfrak{P}$ from 0 to $P$.

In conclusion, we note that Theorems 4 and 5, under the assumption $D_T \in \mathfrak{S}_\infty$, were obtained earlier by the authors in the note ($^3$). For this case, a justification of the above-mentioned formal transformations was carried out by V. M. Brodskii ($^4$). This justification remains valid also in the more general case considered here.

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Received
24 VIII 1966

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* Condition (7) may be replaced by the more general condition

$$\int_{\mathfrak{P}}^{\leftarrow} \left\| |D_T|^{1/2}\, dP\, (I - PD_{TP})^{-1}\, |D_T|^{1/2} \right\| < \infty.$$

*Note: Figure translations are in progress. See original paper for figures.*