INVARIANT SOLUTIONS OF EQUATIONS DESCRIBING THE MOTION OF A LIQUID AND GAS IN LONG PIPELINES

HYDROMECHANICS

1967

SovietRxiv

Abstract

Full Text

UDC 532.542

HYDROMECHANICS

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INVARIANT SOLUTIONS OF EQUATIONS DESCRIBING THE MOTION OF A LIQUID AND GAS IN LONG PIPELINES

(Presented by Academician L. I. Sedov, 22 X 1966)

1°. Consider the system of equations

$$\partial u/\partial t + \partial v/\partial x = 0, \qquad \partial u/\partial x + v^n/u^m = 0. \tag{1}$$

This system covers all practically important cases of the motion of a liquid and gas in a long pipeline. Thus, $m = 1,\ n = 2$ corresponds to turbulent gas flow, $m = n = 1$ to laminar flow. For $m = 0,\ n = 2$ one obtains the equations of motion of a drop liquid. In calculating gas networks one assumes $m = 0$; $1 < n < 2$ depends on the resistance law.

2°. We shall carry out a group classification of equations (1), using the methods developed by L. V. Ovsyannikov ([1,2]).

We investigate whether the system admits an operator

$$X = \xi\,\partial/\partial x + \tau\,\partial/\partial t + \varphi\,\partial/\partial u + \psi\,\partial/\partial v. \tag{2}$$

It is necessary that equations (1) be invariant with respect to the first prolongation of the operator X.

Calculations show that, for general values of m and n, the principal group of system (1) is generated by four independent operators

$$X_1 = \frac{\partial}{\partial x}, \qquad X_2 = \frac{\partial}{\partial t}, \qquad X_3 = x\frac{\partial}{\partial x} + \frac{t}{p}\frac{\partial}{\partial t} - \frac{1}{n}v\frac{\partial}{\partial v},$$

$$X_4 = kx\frac{\partial}{\partial x} + u\frac{\partial}{\partial u} + (k+1)v\frac{\partial}{\partial v}.$$

Here $k = (m - n + 1)/(n + 1)$, $p = n/(n + 1)$.

The finite equations of this group contain 4 parameters a, b, c, d

$$x' = (1 + c + kd)x + a, \qquad t' = (1 + c/p)t + b,$$

$$u' = (1 + d)u, \qquad v' = \{1 - c/n + (k + 1)d\}v.$$

In addition, for $n = 1,\ m = -{}^4/_3$ the operator is possible

$$X_5 = -x^2\partial/\partial x + 3xu\,\partial/\partial u + (xv - 3u^{-1/3})\partial/\partial v.$$

A special case occurs for $n = -1$, when the equations admit an infinite group. The components of the operator (2) contain one or two arbitrary functions and take the following values:

$m \neq -2$:

$$\xi = ax + B(t, u), \qquad \tau = bt + c, \qquad \varphi = \frac{b}{m + 2}u,$$

$$\psi = \left(a - \frac{m + 1}{m + 2}b\right) v + u^{-m}\frac{\partial B}{\partial u};$$

$m = -2:$

$$\xi = A(t, u)x + B(t, u), \quad \tau = bt + c, \quad \varphi = \frac{bt + d}{2}\,u + \frac{b}{2}u \ln u,$$

$$\psi = \left\{A(t, u) + \frac{t + \ln u - 1}{2}b + \frac{d}{2}\right\} v + u^2 x\frac{\partial A}{\partial u} + u^2\frac{\partial B}{\partial u}.$$

The functions $A(t, u)$ and $B(t, u)$ satisfy the equation

$$\frac{\partial}{\partial t} - \frac{\partial}{\partial u}\,u^{-m}\frac{\partial}{\partial u} = 0.$$

We do not consider the heat-conduction equation ($m = 0;\ n = 1$) and the trivial case $n = 0$. When $n = 1$, system (1) describes the phenomenon of nonlinear heat conduction, and the results coincide with those presented in [2].

3°. In what follows, bearing in mind the investigation of nonstationary motion in a long pipeline, we shall restrict ourselves to the study of the group with infinitesimal operators X_1–X_4.

Below is written the optimal system of subgroups θ_1 for the case $k \neq 0$ ($m - n + 1 \neq 0$):

$$X_1,\ X_2,\ X_1+X_2,\ X_4,\ X_2+X_4,\ \alpha X_4+X_3,\ X_1-kX_3+X_4, \tag{3}$$

where α is an arbitrary constant.

Each subgroup from the collection (3), except the first, makes it possible to reduce system (1) to two ordinary differential equations of the first order.

To the operator X_2 there corresponds a solution of the form

$$u=f(x), \quad v=h(x); \tag{4}$$

to the operator X_1+X_2:

$$u=f(x-t), \quad v=h(x-t); \tag{5}$$

to the operator X_4:

$$u=x^{1/k}f(t), \quad v=x^{1+1/k}h(t); \tag{6}$$

to the operator X_2+X_4:

$$u=e^t f(\eta), \quad v=e^{(k+1)t}h(\eta), \quad \eta=e^t x^{-1/k}; \tag{7}$$

to the operator $X_3+\alpha X_4$:

$$u=t^{\alpha p}f(\eta), \quad v=t^{\alpha p(k+1)-1/(n+1)}, \quad \eta=tx^{-1/p(1+k\alpha)}; \tag{8}$$

to the operator $X_1-kX_3+X_4$:

$$u=t^{n/(n-m-1)}f(\eta), \quad v=t^{(m+1)/(n-m-1)}h(\eta), \quad \eta=e^{-x}t^{n/(n-m-1)}. \tag{9}$$

4°. A solution in the form (4) corresponds to stationary motion. Under certain initial conditions one can obtain a solution of the form (5): the initial pressure distribution propagates along the pipe with constant velocity.

It is not difficult to find, by numerical integration, the functions f and h from (6)–(9). However, it is of greatest interest to obtain a solution in elementary functions. We have succeeded in finding such a solution in the form (8) for $\alpha=-(m+1)/(n+1)$:

$$\left(\frac{m+2}{n}\right)^n f^{m-n+1}+k\eta^{-n(n+1)/m+2}=\text{const}, \quad h=-\frac{n}{m+2}\eta^{-n/(m+2)}f.$$

5°. The case $k = 0$ ($m = n-1$) is of special applied interest. The optimal system of one-parameter subgroups consists in this case of the subgroups

$$X_1,\ X_2 + \gamma X_1,\ X_4,\ X_3 + \alpha X_4,\ -X_2 + \beta X_1 + X_4,\ -X_1 + X_4,$$

where α, β, γ are arbitrary constants.

To the operator $X_2 + \gamma X_1$ there corresponds a solution of the form

$$u = f(x - \gamma t), \quad v = h(x - \gamma t); \tag{10}$$

to the operator $X_3 + \alpha X_4$:

$$u = t^{\alpha p} f(\eta), \qquad v = t^{\alpha p - 1/(n+1)} h(\eta), \qquad \eta = x t^{-p}; \tag{11}$$

to the operator $-X_2 + \beta X_1 + X_4$:

$$u = e^{-t} f(x - \beta t), \qquad v = e^{-t} h(x - \beta t); \tag{12}$$

to the operator $-X_1 + X_4$:

$$u = e^{-x} f(t), \qquad v = e^{-x} h(t). \tag{13}$$

6°. The functions f and h in (11) are determined as solutions of the system of ordinary differential equations

$$\alpha f - \eta f' + \frac{1}{p} h' = 0, \qquad f' + \frac{h^n}{f^{n-1}} = 0. \tag{14}$$

Introducing the new unknown function $\zeta = h/f$, we obtain the system

$$\zeta' = -\alpha p - \eta p \zeta^n + \zeta^{n+1}, \tag{15}$$

$$f'/f + \zeta^n = 0. \tag{16}$$

If $\alpha = -1$, it is not difficult to find a particular solution containing an arbitrary constant c:

$$\zeta = p\eta, \qquad f = c \exp\left\{-\frac{p^n}{n+1}\eta^{n+1}\right\};$$

Fig. 1

Figure 1: Fig. 1

Fig. 2

Figure 2: Fig. 2

$$h = cp\eta \exp\left\{-\frac{p^n}{n+1}\eta^{n+1}\right\}.$$

Fig. 1

For turbulent gas flow $n = 2$, this solution in the original variables x, t has the form

$$u = ct^{-2/3}\exp\{-{}^4/_{27}\, x^3 t^{-2}\}, \qquad v = \frac{2c}{3} x t^{-5/3}\exp\{-{}^4/_{27}\, x^3 t^{-2}\}. \tag{17}$$

Solution (17) may be interpreted as a flow in a semi-infinite pipe caused by an explosion-like impulse of the δ-function type.

Fig. 2

The dependence $v(x)$ at times $t_1 < t_2 < t_3$ is shown in Fig. 1. Varying the value of α in (15), we obtain a series of different solutions. For $\alpha = 0$ and $\alpha = 1/n$, one can single out solutions admitting a physical interpretation: in the first case as the propagation of a pressure jump, and in the second as the propagation of a flow-rate jump through initially quiescent gas.

Fig. 2 shows the qualitative behavior of the integral curves of equation (15) for $\eta > 0$ as a function of α.

Solutions with a continuously varying flow rate, defined on the entire positive semi-axis η, admit a physical interpretation. If $a > 0$, the only case of interest for applications is the separatrix of two families of integral curves with vertical asymptotes.

The asymptotic behavior of the separatrix for large η, independently of a, is determined by the formulas

$$\xi \sim p\eta + p^{-n+1}(1+a)\frac{1}{\eta^n} + \cdots,$$

$$f \sim \eta^{-(1+a)n}\exp\left\{-\frac{p^n}{n+1}\eta^{n+1}\right\} + \cdots.$$

For $a < 0$, in addition to the separatrix, an entire family of integral curves is defined for all $\eta > 0$. When η is large, the lines of this family approach the curve

$$\xi = (-a)^{1/n} \eta^{-1/n}.$$

All-Union Scientific Research
Institute of Natural Gases

Received
15 VII 1966

CITED LITERATURE

[1] L. V. Ovsyannikov, DAN, **118**, No. 3 (1958).
[2] L. V. Ovsyannikov, DAN, **125**, No. 3 (1959).

Note: Figure translations are in progress. See original paper for figures.