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Abstract

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MATHEMATICS

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ON THE RECOVERY OF A FUNCTION FROM INTEGRALS OVER ELLIPSOIDS OF REVOLUTION WITH ONE FOCUS FIXED

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In the present article we consider the problem of recovering a function $u(x,s) = u(x_1, \dots, x_n, s)$, even in the variable s, if its integrals are known over confocal ellipsoids of revolution of which one focus is fixed and is located at the origin, while the other runs through all points of the hyperplane $s = 0$. Such a problem is reduced, for example, for even n, to the linearized formulation of the problem of recovering the coefficient $a(x,s)$ of the telegraph equation

$$\frac{\partial^2 w}{\partial t^2} = \sum_{i=1}^{n} \frac{\partial^2 w}{\partial x_i^2} + \frac{\partial^2 w}{\partial s^2} + a(x,s)w + \delta(x,s)\delta(t) \tag{1}$$

for the half-space $s \geq 0$ under the boundary and initial conditions

$$w(x,s,0) = \frac{\partial}{\partial t} w(x,s,0) = \frac{\partial}{\partial s} w(x,0,t) = 0 \tag{2}$$

from the known solution of equation (1) on the hyperplane $s = 0$ under conditions (2) (see ([1])). Here $\delta(x,s)$ and $\delta(t)$ are Dirac delta functions.

Let $S_{x^0,t}$ be the surface of the ellipsoid of revolution

$$r(x,s,0,0) + r(x,s,x^0,0) = t, \tag{3}$$

where $r(x,s,0,0)$ and $r(x,s,x^0,0)$ are, respectively, the distances between the point (x,s) and the points $(0,0)$ and $(x^0,0)$. We shall assume known, for arbitrary x^0 and t, the integrals over the surfaces $S_{x^0,t}$

$$\int_{S_{x^0,t}} u(x,s)\,d\omega = v(x^0,t). \tag{4}$$

Here ω is the solid angle in the space x, s with vertex at the origin.

Theorem 1. *Let the function $u(x,s)$, even in s, be continuous in the variables x, s and satisfy the Hölder condition at the origin; then it is uniquely recovered from the function $v(x^0,t)$.*

The idea of the proof of this theorem consists in finding all moments of the function $u(x,s)$. In the space x, s perform an orthogonal transformation of the unknowns with matrix Q, reducing to a rotation about the origin in the hyperplane $s = 0$. Let under this transformation the point (x,s) pass to the point (y,s), and the point $(x^0,0)$ to the point $(y^0,0)$, where $(y,s) = (y_1, y_2, \dots, y_n, s)$, and $(y^0,0) = (y_1^0, 0, \dots, 0, 0)$. The transformation matrix Q depends on $n-1$ parameters, which may be taken to be the direction cosines $q_1, q_2, \dots, q_{n-1}$ of the radius vector of the point $(x^0,0)$. Introduce spherical coordinates of the point (y,s) by the formulas

$$y_i = r\xi_i, \quad (i = 1,2,\dots,n), \qquad s = r\xi_{n+1}, \tag{5}$$

where ξ_i $(i = 1,2,\dots,n+1)$ are the direction cosines of the radius vector r in the space y, s. The equation of the ellipsoid of revolution can then be written in the form

$$r = p(1-\varepsilon\xi_1)^{-1}, \tag{6}$$

where the parameters ε and p characterize the eccentricity of the ellipsoid of revolution and the polar distance:

$$\varepsilon = y_1^0/t, \qquad p = \tfrac{1}{2}t(1-\varepsilon^2).$$

Formula (4) is then written in the form

$$\int_{S_{q,p,\varepsilon}} u(r\cdot Q\xi)\,d\omega = v(q,p,\varepsilon). \tag{7}$$

Here $\xi = (\xi_1, \xi_2, \dots, \xi_{n+1})$, $q = (q_1, q_2, \dots, q_{n-1})$, and $S_{q,p,\varepsilon}$ is the surface of the ellipsoid of revolution with parameters q, p, ε.

Apply to the left- and right-hand sides of equality (7) the linear operator

$$Lv = p\frac{\partial}{\partial\varepsilon}\int_0^p v(q,z,\varepsilon)\,\frac{dz}{z}, \tag{8}$$

which is meaningful, since the function $u(x, s)$ satisfies Hölder' s condition at the origin. Using formulas (5) and (6), it is not difficult to show the validity of the equality

$$\int_{S_{q,p,\varepsilon}} u(r \cdot Q\xi)\, y_1 \, d\omega = Lv. \tag{9}$$

Applying the operator L repeatedly to equality (7), we similarly obtain, for any natural k, the formula

$$\int_{S_{q,p,\varepsilon}} u(r \cdot Q\xi)\, y_1^k \, d\omega = L^k v. \tag{9'}$$

Let us now note that, by virtue of the orthogonality of the transformation,

$$y_1 \cdot y_1^0 = x_1 \cdot x_1^0 + x_2 \cdot x_2^0 + \cdots + x_n \cdot x_n^0,$$

and since $(x_1, 0)$ is an arbitrary point of the hyperplane $s = 0$, from equality $(9')$ we immediately find the moments of the function $u(x, s)$:

$$\int_{S_{0,t}} u(x, s)\, x_1^{\lambda_1} x_2^{\lambda_2} \cdots x_n^{\lambda_n} \, d\omega = w_k^{\lambda_1, \lambda_2, \ldots, \lambda_n}(t) \qquad (\lambda_1 + \lambda_2 + \cdots + \lambda_n = k),$$

by which the even function $u(x, s)$ in s is uniquely reconstructed.

Let us note that the theorem proved above allows one to assert that, in order to determine the function $u(x, s)$ in some bounded domain D of the space x, s, it is sufficient to know the integrals over ellipsoids that also lie in some bounded domain containing the domain D. Thus, Theorem 1 has a local character.

Below we shall consider the question of reconstructing the function $u(x, s)$ in the case where the integrals of it are given with some weight function. In this case it turns out to be possible also to generalize the form of the surfaces over which the integrals are taken. The theorem obtained in this way will no longer have a local character.

Thus, suppose there is an $(n + 1)$-parameter family of surfaces of revolution $S_{q,p,\varepsilon}$, whose polar equation in the space y, s is written-

is written in the form

$$r = pf(\varepsilon, \varepsilon\xi_1), \tag{10}$$

where $f(\varepsilon, \eta)$ is a function analytic in the variables ε, η in a neighborhood of the origin. Suppose also that the integrals

$$\int_{S_{q,p,\varepsilon}} \varphi(\varepsilon, \varepsilon\xi_1)\, u(r \cdot Q\xi)\, d\omega = v(q, p, \varepsilon), \tag{11}$$

are known, where $\varphi(\varepsilon, \eta)$ is an analytic function of the variables ε, η in a neighborhood of the origin. We shall also assume that $f(0, 0) \neq 0$, $\varphi(0, 0) \neq 0$,

$$\frac{\partial}{\partial\eta}\varphi(\varepsilon, \eta)\Big|_{\eta=0,\ \varepsilon=0} \neq 0.$$

Then the following theorem holds.

Theorem 2. *Let the function $u(x, s)$ satisfy the conditions of Theorem 1 and grow at infinity no faster than a finite power of a logarithm; then it is uniquely determined by the integrals (11).*

The scheme of proof of this theorem is similar to the scheme of proof of Theorem 1; only in this case we shall find the moments of the function $u(x, s)$ for $\varepsilon = 0$, which corresponds to determining the moments on a sphere of radius p.

Expanding the function $\varphi(\varepsilon, \eta)$ in a series in the variable η and using equality (11), we obtain

$$v(q, p, \varepsilon) = \sum_{k=0}^{\infty} \varepsilon^k a_k(\varepsilon) v_k(q, p, \varepsilon), \tag{12}$$

where

$$v_k(q, p, \varepsilon) = \int_{S_{q,p,\varepsilon}} u(r \cdot Q\xi)\, \xi_1^k\, d\omega. \tag{13}$$

From equality (12) we find

$$v_0(q, p, 0) = v(q, p, 0)[a_0(0)]^{-1}.$$

Apply to equality (12) the operator $p^{-1}L$ and expand in the formula

$$p^{-1}Lv_k(q, p, \varepsilon) = \int_{S_{q,p,\varepsilon}} u(r \cdot Q\xi)\xi_1^k\, [f(\varepsilon, \varepsilon\xi_1)]^{-1} \frac{d}{d\varepsilon} f(\varepsilon, \varepsilon\xi_1)\, d\omega$$

the expression

$$[f(\varepsilon, \varepsilon\xi_1)]^{-1} \frac{d}{d\varepsilon} f(\varepsilon, \varepsilon\xi_1)$$

in a series in powers of $\varepsilon\xi_1$. We obtain

$$\begin{aligned} p^{-1}Lv(q,p,\varepsilon) &= c_0(\varepsilon)v_0(q,p,\varepsilon) + \sum_{k=1}^{\infty} \varepsilon^{k-1} c_k(\varepsilon) v_k(q,p,\varepsilon) \\ &\quad + d_0(\varepsilon)\int_0^p v_0(q,z,\varepsilon)\frac{dz}{z} + \sum_{k=1}^{\infty} \varepsilon^{k-1} d_k(\varepsilon)\int_0^p v_k(q,z,0)\frac{dz}{z}, \end{aligned} \tag{14}$$

where $c_1(0) \neq 0$. Letting in this equality the parameter ε tend to zero, we obtain, for determining the function $v_1(q,p,0)$, a Volterra equation of the form

$$v_1(q,p,0) + \lambda \int_0^p v_1(q,z,0)\frac{dz}{z} = f_1(q,p), \tag{15}$$

whose solution, generally speaking, is not unique, since for $\lambda < 0$ it has an eigenfunction of the form $p^{-\lambda}F(q)$, where $F(q)$ is arbitrary

function. However, under the restrictions on the growth of the function $u(x,s)$ at infinity imposed in the hypothesis of the theorem, the function $v_1(q,p,0)$ is found from equation (15) in a unique way.

Applying the operator $p^{-1}L$ to equality (12) k times, we obtain, for determining the function $w_k(q,z) = v_k(q,e^z,0)$, a Volterra integral equation of the form:

$$w_k(q,z) + \int_{-\infty}^{z} \left[\lambda_0(z-\xi)^{k-1} + \lambda_1(z-\xi)^{k-2} + \cdots + \lambda_{k-1}\right] w_k(q,\xi)\, d\xi = f_k(q,z),$$

which, as is not difficult to show by reduction to a differential equation with constant coefficients, has no more than k eigenfunctions whose growth at infinity corresponds to the power growth of the function $v_k(q,p,0)$, and therefore, by virtue of the restrictions imposed on the function $u(x,s)$, the functions $w_k(q,z)$ are all found uniquely.

It is not difficult, analogously to how this was done in the proof of Theorem 1, to find from the functions $v_k(q,p,0)$ the moments of the function $u(x,s)$ on each sphere with center at the origin and thereby determine the function $u(x,s)$.

The following example shows that if even one of the conditions of the theorem is not fulfilled, then, generally speaking, the function $u(x,s)$ cannot be recovered uniquely. Let $n=1$ and let the function $u(x,s)$ have the form

$$u(x,s) = (x^2+s^2)^{\gamma/2} \sum_{k=1}^{m} A_k \cos k\omega,$$

where γ and A_k are arbitrary real numbers. Then there exists such a weight function that the integrals over all ellipses vanish. Indeed,

$$\int_{S^2_{x_0,t}} (1-\varepsilon\cos\omega)^{\gamma} u(x,s)\, d\omega \equiv 0.$$

The indicated nonuniqueness in the reconstruction of the function can be eliminated if its asymptotics at infinity are prescribed.

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Note: Figure translations are in progress. See original paper for figures.