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Abstract

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MATHEMATICS

B. M. BUDAK, A. D. ISKENDEROV

ON A CERTAIN CLASS OF BOUNDARY-VALUE PROBLEMS WITH UNKNOWN COEFFICIENTS

(Presented by Academician A. N. Tikhonov, 16 IX 1966)

In the present note we shall consider boundary-value problems in which, along with the solution of differential equations, certain coefficients of these equations are also unknown. The solvability of the problems considered is ensured by additional boundary conditions. We prove the existence, uniqueness, and stability of the solution of certain problems of this kind, and also estimate the rate of convergence of the method of successive approximations for solving these problems. Earlier, similar boundary-value problems with one unknown coefficient were considered by other methods ([1–3]).

§ 1. A problem of parabolic type.

1°. Suppose it is required to find a triple of functions $\{a(t), c(t), u(x,t)\}$ from the conditions

$$a(t)u_{xx} - c(t)u - u_t = H(x,t,a(t),c(t)), \quad 0 < x < 1, \quad 0 < t \le T; \tag{1}$$

$$u(0,t) = f_1(t), \quad u(1,t) = f_2(t), \quad 0 \le t \le T; \tag{2}$$

$$u(x,0) = \varphi(x), \quad \varphi(0) = f_1(0), \quad \varphi(1) = f_2(0), \quad 0 \le x \le 1; \tag{3}$$

$$-a(t)u_x(0,t) = g(t), \quad g(t) > 0, \quad 0 \le t \le T; \tag{4}$$

$$-a(t)u_x(1,t) = c(t)\psi(t), \quad \psi(t) > 0, \quad 0 \le t \le T, \tag{5}$$

where $H(x,t,a,c)$ is defined, continuous, and bounded in $\overline{\Pi}\{0 \le x \le 1,\ 0 \le t \le T,\ 0 \le a \le A,\ 0 \le c \le C\}$, and the functions $f_1(t)$, $f_2(t)$, $\varphi(x)$, $g(t)$, and $\psi(t)$ are also continuous in their domains of definition.

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Definition 1. A triple of functions $\{a(t), c(t), u(x,t)\}$ will be called a solution of problem (1)–(5) if these functions satisfy the following requirements: 1) $a(t) > 0$ and $c(t) \geq 0$ are continuous for $0 \leq t \leq T$; 2) $u(x,t)$ is continuous in $\overline{D}\{0 \leq x \leq 1,\ 0 \leq t \leq T\}$, while $u_x(x,t)$, $u_{xx}(x,t)$, $u_t(x,t)$ are defined and continuous in $D\{0 < x < 1,\ 0 < t \leq T\}$, and the limits $\lim_{x\to 0+0} u_x(x,t) = u_x(0+0,t)$, $\lim_{x\to 1-0} u_x(x,t) = u_x(1-0,t)$ exist; 3) all relations (1)–(5) are satisfied.

2°. The following existence theorem is valid:

Theorem 1. *Let the following conditions be fulfilled:*

a) $0 < f_{\min} \leq f_1(t) = f(t) \leq f_{\max}$, $f_2(t) \equiv 0$, $\varphi(x) \geq 0$; $\varphi_x(x) \leq 0$, $\varphi_x(0) < 0$, $\varphi_{xx}(x) \geq 0$;

b) $0 \leq H(0,t,a,c) + f_t(t) \leq \nu g_{\min} = \nu \min_t g(t)$, $H(x,t,a,c) \leq 0$, $H_{xx}(x,t,a,c) \leq 0$, $H(1,t,a,c) = 0$, where ν is any number from the interval $0 < \nu < 2$;

c) $\varphi(0) = f_1(0)$, $\varphi(1) = f_2(0) = 0$; $a(0)\varphi_{xx}(0) - c(0)\varphi(0) - f_t(0) = H(0,0,a(0),c(0))$;

c′) $a(0)\varphi_{xxxx}(0) - c(0)\varphi_{xx}(0) - r_t(0,t) = H_{xx}(0,0,a(0),c(0))$,

$$a(0)\varphi_{xxxx}(0) - c(0)\varphi_{xx}(1) = H_{xx}(1,0,a(0),c(0)),$$

where $a(0) = -g(0)[\varphi_x(0)]^{-1}$;

$$c(0) = g(0)\varphi_x(1)[\psi(0)\varphi_x(0)]^{-1},$$

$$r_t(0,0) \equiv [H_t(0,0,a(0),c(0)) + c(0)f_t(0) + f_{tt}(0)][a(0)]^{-1} + [H_a(0,0,a(0),c(0)) + H(0,0,a(0),c(0)) +$$

$$+ f_t(0) - c(0)f(0)][a(0)]^{-2}a_t(0) + [H_c(0,0,a(0),c(0)) + f(0)][a(0)]^{-1}c_t(0),$$

$$a_t(0) = g_t(0)[\varphi_x(0)]^{-1} + g(0)[\varphi_x(0)]^{-2}\{a(0)\varphi^{(0)}_{xxx} - c(0)\varphi_x(0) - H_x(0,0,a(0),c(0))\};$$

$$\begin{aligned} c_t(0) = &\{g_t(0)\varphi_x(1) + g(0)[a(0)\varphi_{xxx}(1) - c(0)\varphi_x(1) - H_x(1,0,a(0),c(0))]\} \\ &\cdot [\psi(0)\varphi_x(0)]^{-1} - g(0)\varphi_x(1)\{\psi_t(0)\varphi_x(0) + \psi(0)[a(0)\varphi^{(0)}_{xxx} \\ &- c(0)\varphi_x(0) - H_x(0,0,a(0),c(0))]\}[\psi(0)\varphi_x(0)]^{-2}; \end{aligned}$$

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d) H, φ, and u_x are differentiable up to the fourth order with respect to x, and also $H_t(x,t,a,c)$, $H_a(x,t,a,c)$, $H_c(x,t,a,c)$, $f(t)$, $f_t(t)$, $f_{tt}(t)$, $g(t)$, $g_t(t)$, $\psi(t)$, $\psi_t(t)$ are continuous and bounded in their domains of definition.

Then there exists a solution of problem (1)–(5) possessing the following differential properties:

$$a(t) \in C[0,T], \qquad c(t) \in C[0,T], \qquad u(x,t) \in C_{2,1}(\overline{D}).$$

The proof is carried out by the method of successive approximations according to the scheme

$$a^{(s)}(t)u_{xx}^{(s)} - c^{(s)}(t)u^{(s)} - u_t^{(s)} = H^{(s)} = H(x,t,a^{(s)}(t),c^{(s)}(t)), \quad 0 < x < 1, \quad 0 < t \le T; \tag{6}$$

$$u^{(s)}(0,t) = f_1(t), \qquad u^{(s)}(1,t) = f_2(t), \qquad 0 \le t \le T; \tag{7}$$

$$u^{(s)}(x,0) = \varphi(x), \qquad \varphi(0) = f_1(0), \qquad \varphi(1) = f_2(0), \qquad 0 \le x \le 1; \tag{8}$$

$$-a^{(s+1)}(t)u_x^{(s)}(0,t) = g(t), \qquad g(t) > 0, \qquad 0 \le t \le T; \tag{9}$$

$$-a^{(s+1)}(t)u_x^{(s)}(1,t) = c^{(s+1)}(t)\psi(t), \qquad \psi(t) > 0, \qquad 0 \le t \le T. \tag{10}$$

Here the following lemmas are used:

Lemma 1. Let the following conditions be satisfied:

a) $0 \le f_{\min} \le f(t) \le f_{\max}$, $\varphi(x) \ge 0$, $\varphi_x(x) \le 0$, $\varphi_{xx}(x) \ge 0$, $\varphi_x(0) < 0$;

b) $a_1(t) \in C_1[0,T]$, $c_1(t) \in C_1[0,T]$, $0 < \alpha_1 = \text{const} \le a_1(t) \le A = \text{const}$, $0 \le c_1(t) \le C = \text{const}$;

c) $-a_1(0)\varphi_x(0) = g(0)$, $-a_1(0)\varphi_x(1) = c_1(0)\psi(0)$;

d) $a_1(t)f_{\max} \ge c_1(t)\psi_{\min}$, $\psi_{\min} = \min_t \psi(t) > 0$;

e) $0 \le H(0,t,a,c) + f_t(t) \le \nu g_{\min}$, where ν is any number from the interval $0 < \nu < 2$, $H(x,t,a,c) \le 0$, $H_{xx}(x,t,a,c) \le 0$, $H(1,t,a,c) = 0$;

f) $a_1(0)\varphi_{xx}(0) - c_1(0)\varphi(0) - f_t(0) = H(0,0,a(0),c(0))$, $\varphi_{xx}(1) = 0$;

g) H, H_x, H_{xx}, φ, φ_x, φ_{xx}, f, f_t are continuous and bounded in their domains of definition.

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Then the solution of the problem

$$a_1(t)u_{1xx} - c_1(t)u - u_{1t} = H(x,t,a_1,c_1), \qquad 0<x<1, \qquad 0<t\le T; \quad (11)$$

$$u_1(0,t) = f(t), \qquad u(1,t)=0, \qquad 0\le t\le T; \quad (12)$$

$$u_1(x,0) = \varphi(x), \qquad 0\le x\le 1, \qquad \varphi(0)=f(0), \qquad \varphi(1)=0, \quad (13)$$

exists continuously and has continuous derivatives u_{1x}, u_{1xx}, and u_{1t} in $\overline{D}\{0\le x\le 1, 0\le t\le T\}$, and the following estimates hold:

$$0\le u_1(x,t)\le Q = f_{\max} + T\max_{\overline{\Pi}}|H|; \quad (14)$$

$$0\le u_{1xx}(x,t)\le B = \max_x \varphi_{xx}(x) + f_{\max}^2\psi_{\min}^{-1} + T\max_{\overline{\Pi}}|H_{xx}| + \nu g_{\min}\alpha_1^{-1}; \quad (15)$$

$$0< f_{\min}\le -u_x(0,t)\le B' = f_{\max} + 2^{-1}B, \qquad 0\le -u_x(1,t)\le f_{\max}. \quad (16)$$

Lemma 2. Under the conditions of Theorem 1, $a^{(s)}(t)\in C_1[0,T]$, $c^{(s)}(t)\in C_1[0,T]$, and $a_t^{(s)}(t)$, $c_t^{(s)}(t)$ are uniformly bounded for all $s=0,1,2,...$ and all t, $0\le t\le T$.

With the help of Lemmas 1 and 2, the compactness in C of the families $a^{(s)}(t)$, $c^{(s)}(t)$, $u^{(s)}(x,t)$, $u_x^s(x,t)$, $u_{xx}^s(x,t)$, and $u_t^s(x,t)$, and the existence of a solution $\{a(t),c(t),u(x,t)\}$ of problem (1)–(5) are established directly; moreover $a(t)\in C[0,T]$, $c(t)\in C[0,T]$, $u(x,t)\in C_{2,1}(\overline{D})$.

3°. Let, alongside problem (1)–(5), there be given the problem ($\bar{1}$)—($\bar{5}$), differing from problem (1)–(5) in that $a,c,u,H,\varphi,f_1,f_2,g,\psi$ are replaced by $\bar a,\bar c,\bar u,\bar H,\bar\varphi,\bar f_1,\bar f_2,\bar g,\bar\psi$. Put

$$\delta_1 = \bar H - H, \quad \delta_2 = \bar\varphi - \varphi, \quad \delta_3 = \bar f_1 - f_1, \quad \delta_4 = \bar f_2 - f_2,$$

$$\delta_5 = \bar g(t) - g(t), \quad \delta_6 = \bar\psi - \psi; \quad (17)$$

$$z(x,t) = \bar u(x,t) - u(x,t), \quad \lambda(t) = \bar a(t) - a(t), \quad \mu(t) = \bar c(t) - c(t). \quad (18)$$

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We shall assume that $\delta_i,\ i = 1, 2, \dots, 6$, are continuous together with the derivatives $\delta_{2x}, \delta_{2xx}, \delta_{3t}, \delta_{4t}$; such perturbations of the data of the problem will be called **admissible**.

Definition 2. If for every $\varepsilon > 0$ there exists a $\delta = \delta(\varepsilon) > 0$ such that, for any admissible $\delta_i,\ i = 1, \dots, 6$, satisfying the conditions

$$|\delta_i| < \delta, \quad i = 1, \dots, 6, \quad |\delta_{2x}| < \delta, \quad \int_0^1 |\delta_{2xx}|\, dx < \delta, \quad |\delta_{it}| < \delta, \quad i = 3, 4, \tag{19}$$

the inequalities

$$|z(x,t)| < \varepsilon \text{ in } \overline{D}, \quad |\lambda(t)| < \varepsilon, \quad |\mu(t)| < \varepsilon \text{ on } [0,T], \tag{20}$$

hold, then we shall say that the solution of problem (1)–(5) is **stable with respect to admissible perturbations of its data**.

Definition 3. We shall say that the solution of problem (1)–(5) belongs to the class $C_{0,0,2,1}$ if $a(t) \in C[0,T]$, $c(t) \in C[0,T]$, $u(x,t) \in C_{2,1}(\overline{D})$.

Theorem 2. *If H, H_a, H_c are continuous in $\overline{\overline{\Pi}}$, then the solution of problem (1)–(5) is unique and stable with respect to admissible perturbations of its data in the class of solutions $C_{0,0,2,1}$.*

From the uniqueness of the solution of problem (1)–(5) it follows that the entire sequence $\{a^{(s)}(t), c^{(s)}(t), u^{(s)}(x,t)\}$ converges to its solution.

Theorem 3. *If, in addition to the conditions of Theorem 1, the conditions*

$$|H_{axx}| \leq \text{const} < +\infty, \quad |H_{cxx}| \leq \text{const} < +\infty, \tag{21}$$

are also satisfied, then the sequence $\{a^{(s)}(t), c^{(s)}(t), u^{(s)}(x,t)\}$ converges to the solution $\{a(t), c(t), u(x,t)\}$ of problem (1)–(5) at the rate of a geometric progression.

§ 2. **Problem for an ordinary differential equation.**

1°. Suppose it is required to find the numbers a, c and the function $u(x)$ from the conditions

$$a u_{xx}^{(x)} - c u(x) = H(x, a, c), \quad 0 < x < 1; \qquad u(0) = f_1,\ u(1) = f_2; \tag{22}$$

$$-a u_x(0+0) = g, \quad g > 0; \qquad -a u_x(1-0) = c\psi, \quad \psi > 0,$$

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where $H(x,a,c)$ is a given continuous function in $\overline{\Pi}_1\{0 \le x \le 1,\ 0 \le a \le A,\ 0 \le c \le C\}$, and f_1, f_2, g, and ψ are given constants.

Definition 4. The triple of quantities $\{a, c, u(x)\}$ will be called a **solution of problem (1)–(4)** if: 1) $a > 0,\ c \ge 0$; 2) $u(x)$ is defined and continuous in $\overline{D}\{0 \le x \le 1\}$, $u_x(x), u_{xx}(x)$ are continuous in $D\{0 < x < 1\}$, and the limits $\lim_{x\to 0+0} u_x(x) = u_x(0+0)$, $\lim_{x\to 1-0} u_x(x) = u_x(1-0)$ exist; 3) all conditions (1)–(4) are satisfied.

2°. The following existence theorem is valid:

Theorem 4. *Let the conditions be satisfied:*

a) $f_1 = f > 0,\ f_2 = 0,\ H(x,a,c) \le 0,\ H_{xx}(x,a,c) \le 0,\ H(0,a,c) = 0,\ H(1,a,c) = 0,\ \max_{\overline{\Pi}_1} |H| \le \nu g,\ f < \nu\psi$, where $\nu > 0$ is any number satisfying the inequality $1 - \nu - 4^{-1}\nu^2 > 0$, i.e. $0 < \nu \le 0.8$; b) H, H_x, H_{xx}, H_a, H_c are continuous and bounded.

Then the solution $\{a, c, u(x)\}$ of problem (1)–(4) exists.

The proof of the theorem is carried out by the method of successive approximations according to the scheme

$$\begin{gathered} a^{(s)} u^s_{xx}(x) - c^{(s)} u^{(s)}(x) = H^{(s)} = H(x, a^{(s)}, c^{(s)}), \quad 0 < x < 1; \\ u^{(s)}(0) = f, \quad u^{(s)}(1) = f_2; \\ -a^{(s+1)} u^{(s)}_x(0) = g, \quad g > 0; \quad -a^{(s+1)} u^{(s)}_x(1) = c^{(s+1)}\psi, \quad \psi > 0. \end{gathered} \tag{23}$$

In doing so, the following lemma is used:

Lemma 3. Let $0 < a_1 \le a_1 \le A,\ 0 \le c_1 \le C$, and suppose $a_1 f \ge c_1\psi$, and let:

a) $f_1 = f > 0,\ f_2 = 0,\ H(x,a,c) \le 0,\ H_{xx}(x,a,c) \le 0,\ H(0,a,c) = H(1,a,c) = 0$;

b) $H,\ H_x,\ H_{xx},\ H_a,\ H_c$ be continuous and bounded. Then for the solution of the problem

$$a_1 u_{1xx} - c_1 u_1 = H(x, a_1, c_1); \qquad u_1(0) = f, \quad u_1(1) = 0 \tag{24}$$

the estimates

$$\begin{gathered} 0 \le u_1(x) \le Q \le 2f + A^{-1} \max_{\overline{\Pi}1} |H(x,a,c_1)|; \quad 0 \le u_{1xx}(x) \le B = \\ = a_1^{-1}\left[CQ + \max_{\overline{\Pi}1} |H_{xx}(x,a_1,c_1)|\right], \end{gathered} \tag{25}$$

$$0 < f \le -u_x(0) \le B = f + 2^{-1}B; \qquad 0 \le -u_x(1) \le f.$$

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hold.

4°. Suppose that, along with problem (1)–(4), a problem $(\bar{1})$–$(\bar{4})$ is given, differing from problem (1)–(4) only in that the quantities $a, c, u, H, f_1, f_2, g, \psi$ are replaced by the quantities $\bar{a}, \bar{c}, \bar{u}, \bar{H}, \bar{f}_1, \bar{f}_2, \bar{g}, \bar{\psi}$. Put

$$\begin{gathered}\delta_1 = \bar{H} - H, \quad \delta_2 = \bar{f}_1 - f, \quad \delta_3 = \bar{f}_2 - f_3, \quad \delta_4 = \bar{g} - g, \quad \delta_5 = \bar{\psi} - \psi, \\ z(x) = \bar{u}(x) - u(x), \quad \lambda = \bar{a} - a, \quad \mu = \bar{c} - c.\end{gathered} \tag{26}$$

We shall assume that δ_1 is a continuous function of all its arguments.

Definition 5. If for every $\varepsilon > 0$ there exists a $\delta = \delta(\varepsilon) > 0$ such that, for all δ_i, $i = 1, 2, \dots, 5$, satisfying the inequality $|\delta_i| < \delta$, the inequalities $|z(x)| < \varepsilon$ for $0 \le x \le 1$, $|\lambda| < \varepsilon$, $|\mu| < \varepsilon$ hold, then the solution $\{a, c, u(x)\}$ of problem (1)–(4) is called **stable with respect to perturbations of its data**.

Theorem 5. Under the conditions of Theorem 4 and the additional condition

$$\max\left\{\int_0^1 |H_a - u_{xx}|\,dx,\ \int_0^1 |H_c - u|\,dx\right\} \le \min(f, 2^{-1}\psi) \tag{27}$$

the solution of problem (1)–(4) is unique and stable.

From the proved uniqueness of the solution of problem (1)–(4) it follows that the entire sequence $\{a^{(s)}, c^{(s)}, u^{(s)}(x)\}$ converges to the solution $\{a, c, u(x)\}$ of problem (1)–(4).

Theorem 6. If, in addition to the conditions of Theorem 4, condition (27) is also fulfilled, then the sequence $\{a^{(s)}, c^{(s)}, u^{(s)}(x)\}$ converges to the solution $\{a, c, u(x)\}$ of problem (1)–(4) with the rate of a geometric progression.

Moscow State University
named after M. V. Lomonosov

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