ON THE BEHAVIOR OF THE SOLUTION OF THE CAUCHY-POISSON PROBLEM FOR LARGE TIME

HYDROMECHANICS

1967

SovietRxiv

Abstract

Full Text

UDC 532.591

HYDROMECHANICS

E. K. ISAKOVA

ON THE BEHAVIOR OF THE SOLUTION OF THE CAUCHY-POISSON PROBLEM FOR LARGE TIME

(Presented by Academician A. A. Dorodnitsyn on 18 XI 1966)

Let $u(x, y, z, t)$, for $t > 0$, $(x, y, z) \in D(-\infty < x, y < \infty,\ h < z < 0)$, $h < 0$, be a solution of the Laplace equation

$$\Delta u \equiv \partial^2 u/\partial x^2 + \partial^2 u/\partial y^2 + \partial^2 u/\partial z^2 = 0, \tag{1}$$

satisfying the conditions

$$\frac{\partial^2 u}{\partial t^2} + g\frac{\partial u}{\partial z} = -\frac{1}{\rho_0}\frac{\partial P}{\partial t} \quad \text{for } z = 0,$$

$$\partial u/\partial z\big|_{z=h} = 0, \qquad u\big|_{\substack{t=0\\z=0}} = \partial u/\partial t\big|_{t=0} = 0, \tag{2}$$

where $P(x, y, t) = |P_0(x, y)|e^{i\omega t}$, $P_0(x, y)$ is a differentiable finite function equal to zero outside the domain Ω; $g > 0$, $\rho_0 > 0$, $\omega > 0$ are constants.

Problem (1)–(2) has a simple physical meaning. Let D be the domain filled by an ideal incompressible homogeneous fluid with density ρ_0, g the acceleration due to gravity, and P the pressure applied to the surface of the fluid $z = 0$. Then problem (1)–(2) is the Cauchy–Poisson problem, and its solution is the velocity potential of the motion of the fluid.

We shall be interested in the behavior of the solution of problem (1)–(2) as $t \to \infty$.*

Let us first discuss the unique solvability of problem (1)–(2). Denote by H the set of functions u that for any $t \geq 0$ are square-integrable in the domain D, i.e. $u \in \mathcal{L}_2(D)$, and for which $u_t \in \mathcal{L}_2(D)$, $\nabla u \in \mathcal{L}_2(D)$ ($\nabla = \partial/\partial x, \partial/\partial y, \partial/\partial z$).

Theorem 1. *The solution of problem (1)–(2) is unique in H.*

Theorem 2. *In H there exists a solution $u(x,y,z,t)$ of problem (1)–(2) such that: a) for all $t \geq 0$ and $(x,y,z) \in D$ it is bounded; b) for all $z \in [0,h]$ and $t \in [0,T]$, $T > 0$ arbitrary, $u = O(1/\rho)$, $\rho = \sqrt{x^2+y^2} \to \infty$.*

We shall seek the solution of problem (1)–(2) in the form

$$u(x,y,z,t) = \frac{1}{2\pi i}\int_S w(x,y,z,\lambda)e^{\lambda t}\,d\lambda, \tag{3}$$

where S is the contour $\operatorname{Re}\lambda = \lambda_1 > 0$ in the complex λ-plane, and w is the solution of equation (1) satisfying the conditions

$$\frac{\partial w}{\partial z} + \frac{\lambda^2}{g}w = \frac{f(x,y)}{\lambda - i\omega} \quad \text{for } z = 0, \qquad \partial w/\partial z\big|_{z=h} = 0 \tag{4}$$

(here $f(x,y) = -i\omega P_0/\rho_0 g$). Then the following lemma holds for w, by means of which Theorem 2 is proved.

---

* This problem, for very special cases, is considered in ([1]).

Lemma 1. For $\operatorname{Re}\lambda \neq 0$ there exists a constant $C > 0$, depending on f, such that

$$|w| \leq C/|\lambda|^3, \qquad \|w\|_{L_2(D)} \leq C/|\lambda|^3, \qquad \|\nabla w\|_{L_2(D)} \leq C/|\lambda|^2,$$

and, uniformly in $z \leq 0$,

$$\|w\|_{L_2(-\infty<x,y<\infty)} \leq C/|\lambda|^3, \qquad \|\nabla w\|_{L_2(-\infty<x,y<\infty)} \leq C/|\lambda|^2.$$

We note that the Fourier transform $\widetilde{w}(\xi,\eta,z,\lambda)$ of the function w with respect to the variables x,y is represented in the form

$$\widetilde{w} = \tilde{f}\,gK(r,z,h)/(\lambda - i\omega)(\lambda^2 + rgA(r)), \tag{5}$$

where $\tilde{f}$ is the Fourier transform of the function f, $r = \sqrt{\xi^2+\eta^2}$;

$$K(r,z,h) = (e^{rz} + e^{r(2h-z)})/(1+e^{2rh}), \qquad A(r) = (1-e^{2rh})/(1+e^{2rh}).$$

And since, by assumption, f is smooth and finite, $\tilde{f}$, $r\tilde{f}$, $r\,\partial\tilde{f}/\partial r$ belong to $L_2(-\infty < x,y < \infty)$ ([2]), while $\tilde{f}$, $r\tilde{f}$, and $K(r,z,h)$ are bounded. Hence, using formula (5), it is not difficult to obtain Lemma 1.

Theorem 3. There exists a function $v(x, y, z)$ and a constant $C > 0$, depending on f, such that for all $z \in [0, h]$ and $t > 0$

$$|u - ve^{i\omega t}| < C\sqrt{\rho}/t$$

uniformly for $\rho \leq t$.

The function v is called the limiting amplitude; it is the limit of the function $ue^{-i\omega t}$ as $t \to \infty$ and can be written out explicitly. Moreover, it can also be defined independently of the function u.

Theorem 4. The function v is the unique solution of equation (1) satisfying the conditions

$$v_z - \frac{\omega^2}{g} v = f \quad \text{for } z = 0, \qquad v_z|_{z=h} = 0; \tag{6}$$

$$\lim_{\rho\to\infty} \sqrt{\rho}\,(v_\rho + ir_\omega v) = 0, \tag{7}$$

where the number r_ω is a solution of the equation $rgA(r) = \omega^2$.

The fact that v is a solution of equation (1) and satisfies condition (6) follows from the fact that $v = \lim_{\lambda\to i\omega}(\lambda - i\omega)w$, and the Sommerfeld radiation condition (7)$*$ for the function v is a consequence of Lemma 2.

Lemma 2. For the functions $v(x, y, z)$ and $\partial v/\partial\rho$, as $\rho \to \infty$ the following asymptotic representations hold:

$$v \cong -ig\sqrt{\frac{2\pi r_\omega}{\rho}}\,\overline{K}(r_\omega, z, h)e^{-i(r_\omega\rho - \pi/4)} \times \iint_\Omega f(\alpha, \beta)e^{i(\alpha\cos\theta + \beta\sin\theta)}\,d\alpha\,d\beta + O\left(\frac{1}{\rho}\right), \tag{8}$$

$$\frac{\partial v}{\partial\rho} \cong r_\omega g\sqrt{\frac{2\pi r_\omega}{\rho}}\,\overline{K}(r_\omega, z, h)e^{-i(r_\omega\rho - \pi/4)} \times \iint_\Omega f(\alpha, \beta)e^{i(\alpha\cos\theta + \beta\sin\theta)}\,d\alpha\,d\beta + O\left(\frac{1}{\rho}\right),$$

where

$$\rho\cos\theta = x, \qquad \rho\sin\theta = y, \qquad \rho = \sqrt{x^2 + y^2}, \qquad f(\alpha, \beta) = -\frac{i\omega}{\rho_0 g}P_0(\alpha, \beta),$$

$$\overline{K} = K(r, z, h)\Big/ \left.\frac{d(rA)}{dr}\right|_{r=r_\omega}.$$

In our case the function v has the form

$$v(x, y, z) = \iint f(\alpha, \beta) G_0^1 \, d\alpha \, d\beta,$$

where

$$G_0^1(\alpha, \beta, x, y, z, t) = g \int_l r K(r, z, h) J_0(rR) \frac{dr}{rgA(r) - \omega^2},$$

*We note that the radiation condition is sometimes, following ([3]), called the outgoing-phase condition.

$R = \sqrt{(x-\alpha)^2 + (y-\beta)^2}$, $J_0(rR)$ is the Bessel function. The contour of integration l in G_0^l passes along the real line from zero to the point $r_\omega - \varepsilon$ and from the point $r_\omega + \varepsilon$ to infinity; it bypasses the point $r = r_\omega$ along the upper semicircle of radius ε with center at the point $r = r_\omega$, where $\varepsilon > 0$ is any number from $(0, \delta)$, $\delta = \pi/4|h|$. Using the asymptotic representation of the Bessel function as $R \to \infty$, one can obtain asymptotic representations for G_0^l and $\partial G_0^l/\partial R$, by means of which Lemma 2 is proved.

The uniqueness of the function $v(x, y, z)$ is proved with the aid of its expansion

$$v(x, y, z) = \sum_k v_k(z) C_k(x, y) \tag{9}$$

in the eigenfunctions $v_k(z)$ of the operator

$$v_k''(z) - \lambda_k^2 v_k(z) = 0,$$

$$\frac{dv_k}{dz} - \frac{\omega^2}{g} v_k = 0 \quad \text{for } z = 0, \qquad \left.\frac{dv_k}{dz}\right|_{z=h} = 0,$$

where λ_k is determined by the equation

$$\lambda_k \operatorname{th} \lambda_k h = \omega^2/g.$$

In our case the system $\{v_k\}$ is complete and orthonormal, and therefore the representation (9) for $v(x, y, z)$ is unique. In (9) the functions $C_k(x, y)$ are solutions of the equation

$$\partial^2 C_k(x, y)/\partial x^2 + \partial^2 C_k(x, y)/\partial y^2 + \lambda_k^2 C_k(x, y) = 0, \quad k = 0, 1, \dots, \tag{10}$$

in which $\lambda_0 = r_\omega > 0$ and $\lambda_k^2 = -(\operatorname{Im}\lambda_k)^2 < 0$, if $k \neq 0$.

This means that $C_0(x, y)$ is determined by equation (10) and condition (7) uniquely (on this point see, for example, (4)). If $k \neq 0$, then, as is known, the functions $C_k(x, y)$ can be determined uniquely by equation (10) and the boundedness condition at infinity. But, in view of (5), $v = O(1/\sqrt{\rho})$; this means that for $k \neq 0$ $C_k(x, y)$ are indeed bounded at infinity.

Thus, if $f = 0$, then $C_k = 0$, $k = 0, 1, \ldots$, and hence $v \equiv 0$.

Remark 1. If $h = -\infty$, then all the results obtained remain valid; only in this case $K = e^{rz}$, $A(r) = 1$, and the number r_ω in condition (7) is equal to ω^2/g.

Remark 2. If the motion of the fluid is regarded as steady, then (8) gives an asymptotic representation for the velocity potential for large ρ; from this it is not difficult to obtain the form of the perturbed surface, the amplitude of the waves that arise, and the energy expended on the formation of these waves.

The asymptotics of such waves in the special case when Ω is a rectangle with sides parallel to the coordinate axes and P_0 is constant were obtained in (6).

Computing Center
Academy of Sciences of the USSR

Received
2 XI 1966

REFERENCES

1. J. J. Stoker, *Water Waves*, 1959.

2. E. Titchmarsh, *Introduction to the Theory of Fourier Integrals*, Moscow-Leningrad, 1948.

3. N. E. Kochin, *Collected Works*, 2, 1949, p. 277.

4. A. N. Tikhonov, A. A. Samarskii, *Equations of Mathematical Physics*, Moscow, 1953.

5. I. N. Vekua, *Proceedings of the Tbilisi Mathematical Institute*, **12**, 105 (1943).

6. S. S. Voit, DAN, **154**, No. 4 (1964).

Note: Figure translations are in progress. See original paper for figures.