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Abstract

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ON A METHOD OF REGULARIZATION OF SINGULAR PERTURBATIONS

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In studying the asymptotic behavior of solutions of differential equations with small or large parameters, different methods are applied to the same equations, but with different properties. For example, for differential equations with small parameters at the highest derivatives, ideas of the exponential boundary layer are applied (see ([2,3])) if the roots of the characteristic equation have negative real parts. If, however, this condition is not satisfied, then fundamentally different methods for constructing approximate solutions are applied to the same equations (see ([5–9,11,12])).

In the theory of oscillations, methods different from those mentioned above are used for the same purposes (see ([1,4])). The principal difficulty in the asymptotic solution of differential equations with parameters is that the solutions depend on the parameters in at least two ways. The aim of most existing methods is to separate these dependences from one another and to work with each of them separately. In application to partial differential equations these methods encounter fundamental difficulties and achieve their aim only for a very narrow class of problems.

In the present note a new approach is proposed to the study of the asymptotic behavior of the solution of certain problems, taking into account the specific character of the dependence of the solution on the parameters. The method will be presented for the example of the Cauchy problem for systems of the form

$$\varepsilon y' = f(y, x) \tag{1}$$

under the assumption of exponential asymptotic stability, i.e., under the condition that

$$\operatorname{Re} \lambda_i(x) < 0, \tag{2}$$

where $\lambda_i(x)$ are the roots of the characteristic equation

$$\mathrm{Det}\,\|\partial f(\varphi(x), x)/\partial y - \lambda E\| = 0.$$

Here $y = \varphi(x)$ is an isolated root of the degenerate equation $f(y, x) = 0$; ε is a small positive parameter; y and f are n-dimensional vectors; the function $f(y, x)$ is defined and continuous, together with its derivatives, in a certain domain $D:\ 0 \le x \le a,\ |y| \le M < \infty$.

The method will then be applied to an inhomogeneous equation with a turning point, for which condition (2) is not satisfied.

Consider the solution of system (1) on some interval $[0, a]$ $(a > 0)$ under the condition

$$y(0, \varepsilon) = D_0. \tag{3}$$

It is known that the solution of system (1) depends on ε both regularly and irregularly. Suppose that the irregular dependence of the solution on ε is known to us, i.e., the solution of problem (1), (3) is representable in the form

$$y = y(x, \psi(x, \varepsilon), \varepsilon), \tag{4}$$

where the function $t = \psi(x, \varepsilon)$ is known; we shall call it the regularizing function. Let it satisfy the following conditions: 1) $\psi(x, 0) = \infty$ for $x > 0$; 2) it has an inverse function $x = \psi^{-1}(t, \varepsilon)$, regular with respect to t and ε; 3) the expression $\varepsilon\psi'(x, \varepsilon)$ is a regular function of ε, and moreover

$$\lim_{\varepsilon \to 0} [\varepsilon\psi'(x, \varepsilon)] = \psi(x) \ne 0 \quad \text{for } x \in [0, a];$$

4) $\psi(0, \varepsilon) = 0$. (The dependence of the solution on ε, indicated in the third place in (4), is assumed to be regular.)

We shall regard the variable $t = \psi(x, \varepsilon)$ as an equal independent variable of the sought solution, alongside the variable x, i.e., we seek the solution of problem (1), (3) as a function of two independent variables and the parameter ε: $y = y(x, t, \varepsilon)$, and we shall study the asymptotic behavior of the solution in the half-strip $\Omega : 0 \le x \le a,\ 0 \le t \le \infty$. Under these conditions

$$y' = \frac{\partial y}{\partial x} + \frac{\partial y}{\partial t}\psi'(x, \varepsilon),$$

and system (1) takes the form

$$\varepsilon\, \partial y/\partial x + \varepsilon\psi'(x, \varepsilon)\, \partial y/\partial t = f(y, x). \tag{5}$$

We shall solve this system of partial differential equations under the conditions

$$y|_{t=0} = F(x), \qquad F(0) = D_0, \tag{6}$$

where $F(x)$ is an arbitrary sufficiently smooth vector-function.

From the very way in which system (5) is obtained, it is clear that any solution of system (1) is a solution of system (5). The converse is also true: upon replacing t by $\psi(x, \varepsilon)$. (We assume that the sought solution $y = y(x, t, \varepsilon)$ is differentiable in the domain Ω for any ε satisfying the inequality $0 \le \varepsilon \le \varepsilon_0$.)

Denote by $M_0(x_0, t_0, y_0)$ an arbitrary point on the initial manifold (6) ($t_0 = 0$).

Under our assumptions one can formulate an algorithm for obtaining the unique solution of problem (1), (3) from the solution of problem (5), (6):

a) Having obtained the solution of problem (5), (6), replace in it the first integral, which takes along each characteristic the value x_0, by its value along the characteristic issuing from the point $A(0, 0, D_0)$ of the space (x, t, y). This is equivalent to our having chosen an integral surface passing through another initial manifold, generally speaking different from $F(x)$, but still issuing from the point A. Denote the obtained integral surface by $y = \Phi(x, t, \varepsilon)$.

b) Project the line of intersection of the surfaces $y = \Phi(x, t, \varepsilon)$ and $t = \psi(x, \varepsilon)$ onto the plane (x, y). This projection is precisely the solution of problem (1), (3).

The justification of this algorithm follows from the method of solution of problem (5), (6) by the method of characteristics.

When condition (2) is satisfied, we take as the regularizing function $\psi(x, \varepsilon) = x/\varepsilon$, i.e. $t = x/\varepsilon$. Then system (5) assumes the form

$$\varepsilon\, \partial y/\partial x + \partial y/\partial t = f(y, x). \tag{7}$$

We shall seek the solution of system (7) in the form

$$y = y_0(x, t) + \varepsilon y_1(x, t) + ... + \varepsilon^n y_n(x, t) + O(\varepsilon^{n+1}).$$

To determine the functions $y_i(x, t)$, we obtain the systems

$$\partial y_0/\partial t = f(y_0, x); \qquad \partial y_i/\partial t - f_y(y_0, x) y_i = h_i(x, t), \quad i = 1, \dots, n, \tag{8}$$

where $h_1(x, t) = -\partial y_0/\partial x$; $\quad h_2(x, t) = -\partial y_1/\partial x + \frac{1}{2} f_{yy}(y_0, x) y_1^2$, etc. We shall solve systems (8), respectively, under the conditions:

$$y_0|_{t=0} = F_0(x), \qquad F_0(0) = D_0;$$

$$y_i|_{t=0} = F_i(x), \qquad F_i(0) = 0, \quad i = 1, \dots, n.$$

Here $F_i(x)$ are arbitrary sufficiently smooth functions. Under our assumptions one can show that all

$$y_i(x,t) \in C^{(n-i+1)}[\overline{\Omega}], \qquad i = 0, 1, \dots, n,$$

and the following is valid:

Theorem 1. Suppose that the following conditions are satisfied: 1) $f(y,x) \in C^{(n+1)}[D]$; 2) $\operatorname{Re}\lambda_i(x) < 0,\ i = 1, \dots, n$; 3) the initial values D_0 belong to the domain of influence of the isolated root $y = \varphi(x)$, and the roots do not change their multiplicity for $x \in [0,a]$; 4) the values of the zeroth approximation $u_0(x,t)$ belong to the domain D for all $t \geq 0$; 5) the functions $F_i(x) \in C^{(n-i+1)}[0,a]$, $i = 0, 1, \dots, n$.

Then, for sufficiently small values $\varepsilon > 0$, the solution $y(x,\varepsilon)$ of problem (1), (3), together with the approximation constructed by the indicated method,

$$\overline{y}_n(x,t,\varepsilon) = y_0 + \varepsilon y_1 + \dots + \varepsilon^n y_n$$

satisfies the inequality

$$\left|y(x,\varepsilon) - \overline{y}_n(x, x/\varepsilon, \varepsilon)\right| < C\varepsilon^{n+1},$$

where C is independent of x and ε.

Obviously, the theorem is true if the functions $y_i(x,t)$ are taken in accordance with item a) of the algorithm indicated above.

In the proof of Theorem 1, the theorem from (13) is used.

As a second example of applying the method of lifting to a space of higher dimension, let us consider the construction of the asymptotics of the solution of the following problem:

$$\varepsilon^3 y'' + xk(x)y = h(x) \qquad (k(x) > 0,\ h(0) \neq 0), \tag{9}$$

$$y(0,\varepsilon) = D_0, \qquad y'(0,\varepsilon) = D_1. \tag{10}$$

For $\varepsilon = 0$, the limiting solution is obtained in the form of the discontinuous function $w = h(x)/xk(x)$.

It turns out that for arbitrary D_0 and D_1, the solution will not tend to the discontinuous limiting solution as $\varepsilon \to 0$. The initial point must be sufficiently far from the origin of coordinates, and the integral curve must leave it at a certain angle, namely:

$$D_1 = \varepsilon^{-1} b[k(0)]^{-1/3} h(0) + \overline{D}_0; \qquad D_1 = -\varepsilon^{-2} c[k(0)]^{-2/3} h(0) + \frac{\overline{D}_1}{\varepsilon}, \qquad (11)$$

where $\overline{D}_0$ and $\overline{D}_1$ are already arbitrary numbers of order $O(1)$ as $\varepsilon \to 0$ (for brevity we shall assume that they do not depend on ε);

$$b = \sqrt[3]{3}\Gamma(4/3) = \int_0^\infty u_2(t)\,dt, \qquad c = \Gamma(2/3)\sqrt[3]{3} = \int_0^\infty u_1(t)\,dt.$$

Here $u_1(t)$ and $u_2(t)$ are Airy functions,* satisfying the equation

$$u'' + tu = 0 \qquad (12)$$

and the initial conditions $u_1(0) = 1,\ u_1'(0) = 0;\ u_2(0) = 0,\ u_2'(0) = 1$.

If the right-hand side $h(x)$ has a zero at zero of at least first order, then the solution of problem (9), (10) will tend to the limiting one (which will no longer be discontinuous) for arbitrary D_0 and D_1.

The regularizing function for the asymptotic solution of problem (9), (11) is taken in the form $t = \varphi(x)/\varepsilon$,

$$\varphi(x) = \left(\frac{3}{2}\int_0^x \sqrt{\tau k(\tau)}\,d\tau\right)^{2/3}$$

(details are omitted, but we note that in the general case the regularizing function $t = \psi(x, \varepsilon)$ is determined so that the terms of the asymptotic series are bounded together with derivatives up to a certain order). Equation (9) is regularized with the help of this function in order to find the asymptotics of a particular solution. To construct the asymptotics of the solution of the homogeneous equation corresponding to equation (9), we use a certain formalization of Langer' s method (7), which makes it possible to determine the general solution of the homogeneous equation in powers of $\varepsilon^{1/2}$, with two degrees of freedom at each step.

* On Airy functions, see (5, 10).

We seek a particular solution in the form

$$\bar{y}(x, \varepsilon) = \varepsilon^{-1}\bar{y}_-(x, t) + \bar{y}_0(x, t) + \varepsilon\bar{y}_1(x, t) + \cdots + \varepsilon^n \bar{y}_n(x, t) + \cdots,$$

using the following lemma.

Lemma. Let the right-hand side of the equation

$$\psi'' - t\psi = F(t) \tag{13}$$

be an entire function and, as $t \to \infty$, have the asymptotic representation

$$F(t) \sim a_k/t^k + a_{k+3}/t^{k+3} + a_{k+6}/t^{k+6} + \cdots \quad (k \geqslant 0)$$

in the angle $|\arg t| < 2\pi/3$. Then the particular solution

$$\psi(t) = \int_t^\infty K(t,\tau)F(\tau)\,d\tau, \quad (K(t,\tau) = u_2(t)u_1(\tau) - u_1(t)u_2(\tau)) \tag{14}$$

of equation (13) (it is an entire function), as $t \to \infty$, has the asymptotic representation

$$\psi(t) = b_{k+1}/t^{k+1} + b_{k+4}/t^{k+4} + b_{k+7}/t^{k+7} + \cdots$$

in the same angle. The coefficients b_{k+i} are expressed in terms of the coefficients of the asymptotic series for the functions $u_i(t)$ and $F(t)$.

Thus the solution of problem (9), (11) is formally represented in the form of the series

$$y(x,\varepsilon) = \varepsilon^{-1}\bar{y}_-(x,t) + y_0(x,t) + \varepsilon y_1(x,t) + \varepsilon^{3/2}y_{1/2}(x,t) + \cdots$$

$$\cdots + \varepsilon^n y_n + \varepsilon^{n+1/2}y_{n+1/2} + \cdots.$$

Denote

$$y_{\varepsilon n}(x) = \varepsilon^{-1}\bar{y}_- + y_0 + \varepsilon y_1 + \varepsilon^{3/2}y_{1/2} + \cdots + \varepsilon^n y_n + \varepsilon^{n+1/2}y_{n+1/2}.$$

Theorem 2. Let $h(x)$ and $k(x)$ have the required number of derivatives on the interval $[0,a]$, let $k(x) > 0$, $h(0) \neq 0$, and let the function $y_{\varepsilon n}(x)$ be obtained by the method indicated above. Then the function $y_{\varepsilon n}(x)$ is an asymptotic approximation to the solution $y(x,\varepsilon)$ of problem (9), (11), i.e. the inequality

$$\left|y^{(i)}(x,\varepsilon) - y_{\varepsilon n}^{(i)}(x)\right| \leqslant C\varepsilon_{n-1+1}, \quad i = 0, 1,$$

holds, where C does not depend on x or ε.

Remark. All terms for $y_{\varepsilon n}(x)$ are written down with the aid of quadratures and Airy functions.

In ([14]) the method has been applied to the model equation of Lighthill. The proposed method can be used in the asymptotic solution of partial differential equations.

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Note: Figure translations are in progress. See original paper for figures.