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Abstract

Full Text

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Let us consider a system of differential equations

$$dx/dt = \varepsilon X(t, x), \tag{1}$$

where $\varepsilon > 0$ is a small parameter; x, X are n-dimensional vectors; the function $X(t, x)$ is continuously differentiable with respect to x in the domain D for $0 \leq t \leq t^*$ and satisfies, with respect to t, the conditions of Dirichlet' s theorem on expansion in a Fourier series of a function that is nonperiodic on $0 \leq t \leq t^*$. In the domain D, for $0 \leq t \leq t^*$, there exist constants M, λ such that the inequalities

$$|X(t, x)| \leq M, \qquad |\partial X(t, x)/\partial x| \leq \lambda \tag{2}$$

hold.

We shall study the behavior of the solution of (1) on a sufficiently large interval $0 \leq t \leq t^*$, where $t^* < \infty$.

Introduce the operation of averaging on a finite interval:

$$X_0(x) = \frac{1}{t^*} \int_0^{t^*} X(t, x)\, dt, \tag{3}$$

where the integration is carried out with respect to the explicit t, while x is regarded as a parameter.

Taking (2) into account, it follows from (3) that

$$|X_0(x)| \leq M. \tag{4}$$

Define the first approximation by the equations

$$d\xi/dt = \varepsilon X_0(\xi), \qquad \xi(0) = x(0). \tag{5}$$

Let **condition A** be satisfied: $\xi(t)$, together with its neighborhood of radius ρ, does not leave the domain D.

We note that in (^{1}), in the equations of the first approximation, instead of (3) one takes

$$X_0(x) = \lim_{T\to\infty} \frac{1}{T} \int_0^T X(t,x)\,dt, \tag{6}$$

i.e., in (^{1}), in order to study the behavior of the solution of (1) on a finite interval, the existence of the mean on an infinite interval is additionally required.

In those cases when, for system (1), the limit (6) does not exist, one may nevertheless, using (3), satisfy all the conditions of theorem (^{1}); for this it is sufficient to extend $X(t,x)$ to an infinite interval periodically with respect to $[0,t^*]$. Then the mean on the infinite interval coincides with (3). In this sense the averaging operation (3) is more general than (6).

It is easy to show that, in the case when $X(t,x)$ is periodic in t on the interval $[0,t^*]$ with period T, the mean (3) will coincide with the mean over the period T, up to terms of order ε, if T is of order unity and t^* is of order $1/\varepsilon$ or greater.

Let us give examples where operation (3) yields an obviously better result than (6). Consider $X(t,x) = xF(t)$. Set $F(t) = \exp(t - t^*)$. Then the mean (6) does not exist, while the mean (3) is a bounded quantity. Or set

$$F(t) = \begin{cases} \sin^2 t, & \text{for } 0 \le t \le t^*, \\ \sin t, & \text{for } t^* \le t \le \infty, \end{cases}$$

where $t^* \sim 1/\varepsilon$. Then the mean (6) is equal to zero, while the mean (3) is $X_0(x) = x/2$, and it is precisely in this case that the smallness of the averaging error is ensured. Indeed, on the interval of interest to us $[0,t^*]$, $X(t,x) = \sin^2 t$ is a function periodic in t with period $T = 2\pi$. In this case the mean (3) and the mean over the period coincide to within ε, and on the basis of theorem (2) one may assert that for $t^* \sim 1/\varepsilon$ the averaging error will be of order εTM, where $M = \max|X(t,x)|$.

In studying system (1), three cases are possible: 1) the limit (6) does not exist and in (5) the mean (3) is taken; 2) the limit (6) exists, but in (5) the mean (3) is taken; 3) the limit (6) exists and in (5) the mean (6) is taken (case (1)).

Let us first examine case 3). Consider the function

$$\widetilde{X}(t,x) = \int_0^t [X(t,x) - X_0(x)]\,dt. \tag{7}$$

In view of (6), there exists a monotone decreasing function $f(t)$, tending to zero as $t \to \infty$, such that in the domain D

$$|\widetilde{X}(t,x)| \leq t f(t). \tag{8}$$

Similarly, it is easy to show that in the domain D

$$|\partial \widetilde{X}(t,x)/\partial x| \leq t\varphi(t), \tag{9}$$

where $\varphi(t) \to 0$ monotonically as $t \to \infty$.

Introducing $\tau = \varepsilon t$, from (8), (9) we obtain

$$\varepsilon|\widetilde{X}(t,x)| \leq \tau f(\tau/\varepsilon), \qquad \varepsilon|\partial \widetilde{X}(t,x)/\partial x| \leq \tau\varphi(\tau/\varepsilon). \tag{10}$$

If τ is fixed and ε is decreased, then the right-hand sides of inequalities (10) will decrease monotonically. Let $\tau \leq L$. Then, however small the prescribed number $a > 0$ may be, there will always be an ε_0 so small that for $\varepsilon < \varepsilon_0$ the inequalities

$$\sup \tau f(\tau/\varepsilon) < a, \qquad \sup \varphi(\tau/\varepsilon) < a \tag{11}$$

will hold. If a is decreased, then inequalities (11) can be satisfied by decreasing ε.

Obviously, the number a may be bounded above by the inequalities

$$a < \rho, \qquad a < \eta^*/[P(\exp \lambda L - 1) + 1], \tag{12}$$

where ρ is introduced by condition A; $P = 1 + M^*/\lambda$; η^* is introduced below.

For case 3) we prove a theorem on the difference between solutions of (1) and (5) that coincide at the initial instant:

For arbitrarily small ρ, η and arbitrarily large L, one can choose an $\varepsilon_0 > 0$ such that, for $0 < \varepsilon < \varepsilon_0$, in the interval $0 < t < L/\varepsilon$ the inequality

$$|x(t) - \xi(t)| < \eta \tag{13}$$

is valid.

For the proof, represent the solution of (1) in the form

$$x = \xi + \varepsilon \widetilde{X}(t,\xi) + ar, \tag{14}$$

where ξ is a solution of (5); $\widetilde{X}(t, \xi)$ is operator (8) as a function of t, ξ; r is a new variable, $r(0) = 0$. The meaning of a is clear from (11), (12).

Substituting (14) into (1), taking account of (5), (7), and $\tau = \varepsilon t$, it is easy to obtain the integral equation

$$ar = \int_0^\tau \left[X(t, \xi + \varepsilon \widetilde{X} + ar) - X(\xi) - \varepsilon \frac{\partial \widetilde{X}(t, \xi)}{\partial \xi} X_0(\xi) \right] d\tau. \tag{15}$$

We shall find the solution of (15) by means of successive Picard approximations

$$r = r_0 + (r_1 - r_0) + \dots + (r_n - r_{n-1}) + \dots, \tag{16}$$

where $r_0 = 0$,

$$ar_n = \int_0^\tau \left[X(t, \xi + \varepsilon \widetilde{X} + ar_{n-1}) - X(t, \xi) - \varepsilon \frac{\partial \widetilde{X}(t, \xi)}{\partial \xi} X_0(\xi) \right] d\tau, \quad n = 1, 2, \dots \tag{17}$$

Using (2), (4), (10), (11), for $\varepsilon < \varepsilon_0$ we obtain for the terms of the series (16) the majorant estimates

$$|r_1| \leq P\lambda\tau, \dots, \qquad |r_n - r_{n-1}| \leq P\lambda^n \tau^n / n!, \dots \qquad (P = 1 + M/\lambda). \tag{18}$$

The estimates (18) allow us to conclude that for any finite L, where $L = \max \tau$, the series (16) will converge uniformly to r, and that

$$|r| \leq P(\exp \lambda L - 1). \tag{19}$$

Taking account of (12), (14), (19), for $R = x - \xi$ we obtain

$$|R| = |\varepsilon \widetilde{X} + ar| \leq a[P(\exp \lambda L - 1) + 1] < \eta^*, \tag{20}$$

where $\eta^* = \min(\rho, \eta)$.

It is clear from (20) that, for fixed η^* (fixed ρ, η) and decreasing a (decreasing ε), inequality (20) will be satisfied by larger and larger L.

By virtue of condition A and since, obviously, $\eta^* \leq \rho$, it follows from (20) that the right-hand side of (14) belongs to the domain D. Since $\eta^* \leq \eta$, it follows from (20) that $|x - \xi| < \eta$, and the theorem is proved.

This theorem (^{1}), proved by a method different from (^{1}), in connection with the introduced operation (3), is clearly applicable not only to case 3) (case (^{1})), but also to cases 1), 2), in view of the possibility of extending $X(t,x)$ to an infinite interval periodically in t with period t^*. In this case the mean (6) coincides with the mean (3).

Thus, we have established that theorem (^{1}) is valid under less stringent restrictions on $X(t,x)$ with respect to t than in (^{1}), and for another principal part for $X(t,x)$ ((3) instead of (6)).

Let us note the following. The theorems just proved and (^{1}) guarantee that, as ε decreases, the interval on which the averaging error remains less than the prescribed quantity η will expand. For the case when $X(t,x)$ is periodic in t with period $T \ll t^*$, a better result was obtained in (^{2},^{3}): as ε decreases, the averaging error decreases on an expanding interval. In particular, it follows from (^{2},^{3}) that, for $T, M, \lambda, L \sim 1$, the averaging error will be of order ε on an interval of order $1/\varepsilon$, and, conversely, this result does not follow from (^{1}) and the present work.

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CITED LITERATURE

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* With the aid of (3) it is easy to show that replacing the differentiability requirement for $X(t,x)$ with respect to x by Lipschitz conditions leads to an inessential complication of the proof.

Note: Figure translations are in progress. See original paper for figures.