DEFORMATION OF AN ORTHOTROPIC ELASTIC-VISCOUS BODY UNDER CONDITIONS OF A PLANE PROBLEM

THEORY OF ELASTICITY

1967

SovietRxiv

Abstract

Full Text

UDC 539.371

THEORY OF ELASTICITY

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DEFORMATION OF AN ORTHOTROPIC ELASTIC-VISCOUS BODY UNDER CONDITIONS OF A PLANE PROBLEM

This article considers the deformation of fiberglass plastics under conditions of a plane problem. We shall assume that fiberglass plastic is an orthotropic body both with respect to its elastic properties and with respect to its elastic-viscous properties. It is established here that, for a number of plane problems, the determination of the change in deformation over time reduces to the calculation of single quadratures along certain contours. For relatively small times of action of the load, this problem can be reduced to a series of recurrent algebraic operations.

The relations between strains and stresses in an orthotropic elastic-viscous body will be the following:

$$\begin{aligned}
\varepsilon_x &= a_{11}\sigma_x + \int_0^t K_{11}(t-\tau)\sigma_x\,d\tau + a_{12}\sigma_y + \int_0^t K_{12}(t-\tau)\sigma_y\,d\tau,\\
\varepsilon_y &= a_{12}\sigma_x + \int_0^t K_{12}(t-\tau)\sigma_x\,d\tau + a_{22}\sigma_y + \int_0^t K_{22}(t-\tau)\sigma_y\,d\tau, \qquad (1)\\
\gamma_{xy} &= a_{33}\tau_{xy} + \int_0^t K_{33}(t-\tau)\tau_{xy}\,d\tau.
\end{aligned}$$

We shall also assume that this relation is valid both for compression and for tension.

Experimental studies (^{4}, ^{5}) make it possible to conclude that, for some relatively small interval of time, the kernels in expression (1) can be approximated quite satisfactorily by means of power functions. In this case the exponents are relatively close to one another (their values in most cases lie in the interval 0.74–0.78). In what follows they will be assumed equal. Therefore, in relations (1) we take

$$K_{ij}(t-\tau) = k_{ij}(t-\tau)^{\alpha-1}. \qquad (2)$$

In these equations the coefficients characterizing the elastic-viscous properties possess symmetry with respect to the diagonal. This circumstance, at any rate, will occur when the fiberglass fabric on the basis of which the fiberglass plastic is made has a "matting" type weave, with the weft and warp being identical.

In the case of an elastic-viscous material possessing only shear creep,

$$k_{11} = k_{12} = k_{22} = 0, \qquad k_{33} \neq 0.$$

As experiments show, fiberglass plastics in many cases possess precisely this property. In what follows, this particular case will be investigated in greater detail.

Let us note that, in determining the stresses, no additional difficulties arise in the case where as the kernel there is used

expression

$$k(t-\tau)^{\alpha-1}e^{-a(t-\tau)}. \tag{3}$$

If we introduce into consideration the operational transforms of the stresses and strains: $\tilde{\sigma}_x, \tilde{\sigma}_y, \tilde{\tau}_{xy}, \tilde{\varepsilon}_x, \tilde{\varepsilon}_y, \tilde{\gamma}_{xy}$, then, on the basis of (1) and (2), we shall have

$$\begin{aligned} \tilde{\varepsilon}_x &= a_{11}^*\tilde{\sigma}_x + a_{12}^*\tilde{\sigma}_y, \\ \tilde{\varepsilon}_y &= a_{12}^*\tilde{\sigma}_x + a_{22}^*\tilde{\sigma}_y, \\ \tilde{\gamma}_{xy} &= a_{33}^*\tilde{\tau}_{xy}. \end{aligned} \tag{4}$$

Here

$$\begin{aligned} a_{11}^* &= a_{11} + k_{11}\Gamma(\alpha)p^{-\alpha}, \qquad & a_{12}^* &= a_{12} + k_{12}\Gamma(\alpha)p^{-\alpha}, \\ a_{22}^* &= a_{22} + k_{22}\Gamma(\alpha)p^{-\alpha}, \qquad & a_{33}^* &= a_{33} + k_{33}\Gamma(\alpha)p^{-\alpha}. \end{aligned} \tag{5}$$

Relations (4) are equivalent to the relations of Hooke' s law for an orthotropic body. Thus, in order to solve the plane problem of the deformation of a viscoelastic body, it is necessary to solve the problem of deformation of an orthotropic elastic body. However, the determination of the components of strain and stress, i.e., the finding of originals from images, is often associated with considerable difficulties. This is due to the circumstance that, in order to compute the integrals, it is necessary to know all the singularities of the integrand in the plane of the complex variable p.

Below a class of problems will be indicated for which determination of the sought quantities is relatively simple. In this case it proves possible to establish all branch points of the function in the Riemann-Mellin integral and thereby compute its value. Let us note that for an isotropic elastic-viscous body this transition is not complicated (see (3)).

In some cases it proves possible to seek the quantities to be determined in the form of series; however, this is not always possible. Moreover, the indicated method is often insufficiently effective.

Let us investigate the case when the expressions for the stress components can be given in the form of single quadratures.

Let us consider, as an example, the deformation of a fiberglass plate under the action of a force applied to the boundary, which is equal to zero for $t < 0$ and equal to P for $t > 0$. We introduce polar coordinates r and θ.

The images of the stresses arising in the half-plane are determined by the formulas (^{1,2})

$$\tilde{\sigma}_r = \frac{P}{\pi}(u_1 + u_2)\sqrt{a_{11}^* a_{22}^*}\,\frac{\cos\theta}{rL(\theta)}, \tag{6}$$

$$\tilde{\sigma}_\theta = 0.$$

Here

$$L(\theta) = a_{22}^* \sin^4\theta + (2a_{12}^* + a_{33}^*)\sin^2\theta\cos^2\theta + a_{11}^*\cos^4\theta. \tag{7}$$

In addition, u_1 and u_2 are the roots of the equation

$$a_{22}^* u^4 - (2a_{12}^* + a_{33}^*)u^2 + a_{11}^* = 0. \tag{8}$$

In the case where only shear creep takes place,

$$a_{11}^* = a_{11}, \qquad a_{12}^* = a_{12}, \qquad a_{22}^* = a_{22}, \qquad a_{33}^* = a_{33} + k\Gamma(\alpha)p^{-\alpha}. \tag{9}$$

In this case (6) can be transformed as follows:

$$\tilde{\sigma}_r = -\frac{p}{\pi}\sqrt{a_{11}a_{22}}\,q(p)\,\frac{1}{r}\,\frac{1}{F(\theta) + G(\theta)p^{-\alpha}}, \tag{10}$$

where

$$F(\theta) = \frac{1}{\cos\theta}\left(a_{22}\sin^4\theta + (2a_{12} + a_{33})\sin^2\theta\cos^2\theta + a_{11}\cos^4\theta\right),$$

$$G(\theta) = k\Gamma(\alpha)\sin^2\theta\cos\theta.$$

Moreover,

$$q(p) = u_1 + u_2 = c\left\{\sqrt{(A+Bp^{-\alpha}) + \sqrt{A+Bp^{-\alpha})^2 - 1}} + \sqrt{(A+Bp^{-\alpha}) - \sqrt{(A+Bp^{-\alpha})^2 - 1}}\right\},$$

$$A = (2a_{12} + a_{33})/2\sqrt{a_{11}a_{22}}, \qquad B = k\Gamma(\alpha)/2\sqrt{a_{11}a_{22}}, \qquad C = (a_{11}/a_{22})^{1/4}. \tag{11}$$

The only branch point of the expression $q(p)$ will be $p = 0$, since for real $p > 0$ the quantity $(A + Bp^{-\alpha})^2 - 1$ will be positive; this is a consequence of the fact that in the present case, for an orthotropic body, $A > 1$, and the roots u_1 and u_2 are real.

To determine how the radial stress changes with time in comparison with the instantaneously established value σ_r^0, we shall consider the quantity

$$\sigma_r - \sigma_r^0 = -\frac{P}{\pi}\sqrt{a_{11}a_{22}}\frac{1}{r}\left\{\frac{q(p)}{F(\theta) + G(\theta)p^{-\alpha}} - \frac{q_0\cos\theta}{L(\theta)}\right\},$$

$$q_0 = c\left[\sqrt{A + \sqrt{A^2 - 1}} + \sqrt{A - \sqrt{A^2 - 1}}\right]. \tag{12}$$

In this case

$$\sigma_r - \sigma_r^0 = \frac{1}{2\pi i}\int_{\gamma - i\infty}^{\gamma + i\infty} \frac{e^{pt}}{p}\left(\tilde{\sigma}_r - \sigma_r^0\right) dp.$$

In the present case, in view of the above-mentioned singularities of the integrand, the contour of integration may be deformed; the integration must be performed along the path from $-\infty$ to 0 for $p = \xi - i0$ and from 0 to ∞ for $p = \xi + i0$, bypassing the origin (here $p = \xi + i\eta$). If we set $p = \rho e^{i\theta}$, we shall finally have

$$\sigma_r - \sigma_r^0 = -\frac{P}{\pi}\sqrt{a_{11}a_{22}}\frac{2}{r}\int_0^\infty \frac{e^{\rho t}}{\rho}\left\{\operatorname{Im}\left(\frac{q(\rho^{-\alpha}e^{-\alpha\pi i})}{F(\theta) + G(\theta)\rho^{-\alpha}e^{-\alpha\pi i}}\right) - \frac{q_0\cos\theta}{L(\theta)}\right\} d\rho. \tag{13}$$

It is not difficult to find the imaginary part of the expression under the integral sign.

In the preceding discussion a kernel of the form (2) was used. If one uses the kernel (3), suitable for describing processes whose duration is arbitrarily long, and also considers the case in which only shear creep takes place, then one of the expressions (9) takes the form

$$a_{33}^{*} = a_{33} + k\Gamma(\alpha)(p+\theta)^{-\alpha}. \tag{14}$$

Thus, in formula (10), $p^{-\alpha}$ must be replaced by $(p+a)^{-\alpha}$. The evaluation of the contour integral in this case also presents no difficulty. However, the character of the solution obtained (its tendency to an asymptotic value as $t \to \infty$) will be especially clear if one uses the following, easily proved relation:

$$\sigma_r^{(1)}(t) - \sigma_r^0 = e^{-at}(\sigma_r(t) - \sigma_r^0) + a\int_0^t e^{-a\xi}(\sigma_r(\xi) - \sigma_r^0)\,d\xi. \tag{15}$$

Here $\sigma_r^{(1)}(t)$ is the value of the radial stress in the case when the kernel (3) is used.

Expressions in the form of a single quadrature can also be obtained in the following cases: stretching of a plate with an elliptical hole, the action of a punch rigidly connected to a fiberglass plate, and the deformation of a fiberglass strip. In all these cases it proves possible to determine the branch points of the integrand in the Riemann–Mellin integral and to obtain formulas analogous to (13).

Let us note that the problem becomes considerably simpler for relatively small times (this smallness depending on the characteristics of viscoelasticity). Using certain relations pertaining to the Laplace transform, one can show that in this case it is sufficient to restrict oneself to finding the asymptotic value of the original function for large values of p; in other words, it is necessary to know the expansion in a neighborhood of the infinitely distant point, which in most cases presents no difficulty.

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Received
28 VII 1967

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Note: Figure translations are in progress. See original paper for figures.

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