A linear parabolic problem in a cylinder, infinite with respect to time. Stabilization of solutions

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Abstract

Full Text

Preamble

DIFFERENTIAL EQUATIONS

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A LINEAR PARABOLIC PROBLEM IN AN INFINITE CYLINDER: STABILIZATION OF SOLUTIONS

We consider a general linear parabolic problem in an infinite cylinder $Q = G \times R^+$, where G is a finite n-dimensional domain in space bounded by an $(n-1)$-dimensional surface Γ. We assume Γ is infinitely smooth, locally rectifiable, and oriented. The time domain R^+ is the semi-infinite line $t > 0$. The problem is formulated as follows:

$$\begin{aligned}\frac{\partial u}{\partial t} &= A(x, \frac{\partial}{\partial x})u + f(x,t), \quad (x,t) \in Q \\ B(x,t)u &= \phi(x,t), \quad (x,t) \in S \\ u|_{t=0} &= u_0(x), \quad x \in G\end{aligned}$$

Here, u and f are vector columns of height m, and $S = \Gamma \times R^+$ represents the lateral surface of the cylinder. The operator A is a square matrix differential operator of the form:

$$A(x, \frac{\partial}{\partial x}) = \sum_{|k| \leq 2p} a_k(x) D^k$$

where $k = (k_1, k_2, \dots, k_n)$ is a multi-index, $|k| = \sum k_i$, and $D^k = \frac{\partial^{|k|}}{\partial x_1^{k_1} \dots \partial x_n^{k_n}}$. The boundary conditions are defined by the operator $B(x,t)$, which typically involves derivatives with respect to the spatial coordinates.

The primary objective of this study is to investigate the behavior of the solution $u(x,t)$ as $t \to \infty$. Specifically, we analyze the conditions under which the solution stabilizes, meaning it approaches a steady-state or follows a predictable asymptotic trajectory as time increases indefinitely. This involves examining the spectral properties of the operator A and the regularity of the source terms $f(x,t)$ and boundary data $\phi(x,t)$.

The system $\frac{\partial u}{\partial t} = \dots$ satisfies the condition of parabolicity in the sense of I. G. Petrovskii (see [**?**]) in the domain $Q = \Omega \times (0,T)$, where $T > 0$. Specifically, the principal part of the matrix D satisfies the condition $\det A(x,t,\xi) \neq 0$ for $|\xi| \neq 0$. Here, $2m$ denotes the order of the system (1), p represents the parabolic weight (which is an integer), and $B(x,t)$ is a rectangular matrix satisfying the "regular solvability" condition (see [**?**, **?**, **?**, **?**, **?**]). Specifically, this refers to a one-dimensional boundary value problem involving matrices, where the principal parts of the matrices $A(x,t)$ and $B(x,t,p)$ are given by:

$$A(x,t)\frac{\partial^2 z}{\partial x^2} + B(x,t,p)\frac{\partial z}{\partial x} + \dots$$

written in the local coordinate system, is uniquely solvable. The functions $a(x,t)$ and $b(x,t)$ are infinitely smooth in Q. Under these assumptions, M. S. Agranovich and M. I. Vishik (see [**?**]) studied the parabolic problem in a finite cylinder with operators $A(x,t)$ and $B(x,t)$ of general form:

$$\begin{aligned} &A(x,t)u = f(x,t) \text{ in } Q_{0T} \\ &B(x,t)u \to g(x',t) \text{ on } Q'_{0T} \end{aligned}$$

where $Q_{0T} = G \times (0 < t < T)$ and $Q'_{0T} = \Gamma \times (0 < t < T)$. They also examined the parabolic problem in an infinite cylinder for the case where the coefficients $a(x,t) = a(x)$ and $b(x,t) = b(x)$ depend only on x.

In the aforementioned work, the problem (7)-(9) is investigated by reducing it to a stationary (semi-bounded) problem.

LINEAR PARABOLIC PROBLEM IN A CYLINDER INFINITE IN TIME

Along with equations (1)-(3), we will consider the problem:

$$\begin{aligned} &A(x,D)u = A(x,D+p)u = f(x), \quad (x,t) \in Q_{\pm} \\ &B(x,D)u = B(x,D+p)u = g(x), \quad t \to +0 \end{aligned}$$

This is obtained from (1)-(3) by substitution. In particular, $A(x)u \sim A(x, D+p)u = f(x)$.

For $(x', t) \in Q_+$, as $t \to +0$ and $p \to +0$ with $p < -1$ and $x \in G$, we will primarily employ the notation established in works [**?**, **?**, **?**, **?**]. Let l be an arbitrary fixed number satisfying the condition that l is an integer such that $l > \max(2m, m_j + 1)$. We define $\mu = l - 2m$ and $\mu_j = l - m_j - \frac{1}{2}$ for $x \in G$.

The space is defined as $\mathcal{H}^l(Q_{T_1T_2}, Q'_{T_1T_2}) = H^l(Q_{T_1T_2}) \times \prod_{j=1}^{m} H^{\mu_j}(G)$. Here, $H = W_2^l$ denotes the Sobolev space (see [**?**]) and H_μ denotes L. N. Slobodetskii spaces (see [**?**] and [**?**]). The norms of the vectors $(f, g) \in (G; \Gamma)$ are defined as the sum of the norms of their components:

$$\|(f,g)\|_{(G;\Gamma)} = \|f\|_{W_p^s(G)} + \sum \|g_j\|_{W^{s-1/p}p(\Gamma)}$$

Let $\mathcal{A}$, $\mathcal{A}_j$ $(j = 1, 2, 3)$, and $\mathcal{A}_4$ be the operators corresponding to problems (1)-(3), (7)-(9), (12)-(14), and (15)-(17), respectively, acting on the appropriate spaces. Let $\mathcal{L}$ be the operator corresponding to problem (4)-(6), acting from the subspace of the space H consisting of functions consistent with zero at $t = 0$, into the subspace of the space of functions (f, g) consistent with zero at $t = 0$. Finally, let Π be the operator:

$$\Pi(x) : w \to (A(x)w, B(x)w|_{x'}) \in G^{(p)}$$

corresponding to problem (10), (11). Estimates related to functions belonging to $H(e^{-pt})$ or H will be written with constants that are independent of the functions themselves and the parameter p. We present here the formulation of the main results from the cited work by M. S. Agranovich and M. I. Vishik:

Theorem I. For a finite T, the operator $\Pi(x,t)$ possesses an inverse $\Pi^{-1}(x,t)$ acting from the subspace of elements g in H that are compatible with zero at $t = 0$ to the subspace of elements u in H that are also compatible with zero at $t = 0$. Moreover, the following estimate holds:

$$C(\gamma)\|\Pi(x,t)u\| \leq \|u\| \leq C'(\gamma)\|\Pi(x,t)u\|$$

Theorem II. For each operator $\Pi(x)$, one can specify a number $\delta = \delta(\Pi(x))$, depending only on $\Pi(x)$, such that:

a) For any p satisfying the inequality $\operatorname{Re} p > \delta$, the operator $\Pi(x)$ possesses an inverse $\Pi^{-1}(x)$ acting from $H_s(G)$ to $H_{s+2b}(G)$, and the following estimate holds:

$$C_1\|u\|_{H_{s+2b}(G)} \leq \|\Pi(x)u\|_{H_s(G)} \leq C_2\|u\|_{H_{s+2b}(G)}$$

b) For any p satisfying the inequality $\mathrm{Re}\, p > \delta$, the operator $\Pi(x)$ possesses an inverse acting on $H(e^{-pt})(Q_+;\Omega_+)$, and the following estimate holds:

$$\|u\|_{H(e^{-pt})(Q_+;\Omega_+)} \leq C\|\Pi(x)u\|_{H(e^{-pt})(Q_+;\Omega_+)}$$

We aim to extend these results to the general case of operators $\Pi(x,t)$ and to investigate the stabilization property of solutions as $t \to +\infty$.

EXISTENCE AND UNIQUENESS THEOREM. APRIORI ESTIMATES

In this section, we assume that the derivatives of the functions $b(x,t)$ satisfy the following condition: for any $\epsilon > 0$, one can specify a $T(\epsilon)$, depending only on the fixed ϵ, such that for $t > T(\epsilon)$ and $x \in G$:

$$|a_{ij}(x,t) - a_{ij}(x)| < \epsilon \tag{24}$$

The Lemmas I-V presented in this section will be proven using a method entirely analogous to the proofs of the corresponding propositions in work [**?**].

Lemma I

The operator $L(x,t)$ (corresponding to problem (1)-(3)) acts boundedly from $H^{l+2,\frac{l+2}{2}}(Q_\infty)$ to $H^{l,\frac{l}{2}}(Q_\infty)$ for any ρ satisfying the specified conditions. More precisely, the following estimate holds:

$$\|e^{-\rho t}L(x,t)u\|_{H^{l,\frac{l}{2}}(Q_\infty)} \leq C\|e^{-\rho t}u\|_{H^{l+2,\frac{l+2}{2}}(Q_\infty)} \tag{26}$$

We now consider the following problem in the domain $\Omega_\Lambda(x,\tau+T)$:

$$u = A(x,\tau+T)w = f(x,\tau), \quad (x,\tau) \in \Omega$$
$$B(x,\tau+T)u = \phi(x,\tau), \quad (x,\tau) \in \Omega^+$$

This corresponds to the operator $\Pi(T) : u \to (A(x,x+T)u|_{Q(T)}; B(x)u|_{Q(T)})$. For any $T > 0$, an estimate analogous to (26) holds:

$$\|u - \Pi(x,x+T)u\| \leq C\|u\| \tag{26'}$$

Lemma II

For any $\epsilon > 0$, there exists a $T(\epsilon)$ depending only on ϵ such that for $T > T(\epsilon)$, the operator $\Pi(x,x+T) - \Pi(x) : u \to ((A(x,x+T) - A(x))u; (B(x,x+T) - B(x))u)$ satisfies:

$$\|\Pi(x,x+T) - \Pi(x)\| \leq \epsilon\|e^{-\beta T}u\| \tag{26''}$$

Lemma III

There exists a T_0 such that for $T \geq T_0$, the operator $A(x, \tau + T)$ satisfies the following a priori estimate:

$$\|u\|_{W_2^{2b,\tau}(\Omega)} \leq 2C_1 \|e^{-p\tau} A(x, \tau + T) u\|_{L_2(Q)} \tag{34}$$

Lemma IV

There exists a T_0 such that for $T \geq T_0$, the operator $A(x, \tau + T)$ possesses a bounded inverse $A^{-1}(x, \tau + T)$. **Proof.** According to Theorem II, b) of M. S. Agranovich and M. I. Vishik, the operator $A(x)$ for Re $p > \delta$ possesses an inverse acting as a bounded operator. We write $A(x, \tau+T) = A(x) + (A(x, x+T) - A(x))$. Multiplying on the right by $A^{-1}(x)$, we obtain $A(x, \tau+T)A^{-1}(x) = I + S(x, \tau + T)$, where $S(x, \tau + T) = (A(x, x + T) - A(x))A^{-1}(x)$. By virtue of (26’) with $\epsilon = \frac{1}{4}$, the operator $I + S(x, \tau + T)$ possesses a bounded inverse.

Lemma V

For Re $p > \sigma$, where $\sigma = \sigma(\Lambda(x))$, the following a priori estimate holds for the operator $\Lambda(x, T + \tau)$:

$$\|(I - P)u\|_H \leq C \|\Lambda(x, T + \tau)u\|_H$$

SPATIAL STABILIZATION OF SOLUTIONS

According to the theory developed by L. N. Slobodetskii (see [**?**]), one can construct a continuation into the domain $\Omega_{T,\infty}$. Denoting this continuation by $\tilde{u}$, it satisfies:

$$\|\tilde{u}\|_H \leq C e^{\rho(t-T)} \|f\|_H, \quad T \in \Omega_\infty$$

It is clear that $e^{P(t-T)}$ holds, and by applying the substitution $x = t - T$, we obtain $(x + T)(u - u_{app})e^{P(t-T)}$. Then, according to (34), for $T > T_0$ and Re $p > \delta$, we have:

$$\|(x, x + T)(u - u_{app})e^{-P(t-T)}\| < \text{const } R \|e^{-P(t-T)} \Pi(x, t) u_{app}\|$$

Theorem. For any p satisfying the condition Re $p > \delta$, the operator $L(p)$ corresponding to problem (1)-(3) possesses an inverse $L^{-1}(p)$ acting in H_s, and the following estimate holds:

$$\|u(x, t)\| \leq C \|e^{-\mu t} f(x, t)\| \tag{38}$$

Lemma. Let $\alpha > 0$ be a constant. Then the function $z = \int_0^x \frac{f(t)}{(1+t)^\alpha} dt$ possesses the property that for any constant $\epsilon > 0$, the limit exists in the L_p metric.

Lemma. Let condition (45) be satisfied for some constant. Then the solution to the problem (15)-(17) and the solution v to the problem (7)-(9) possess the property that for any constant $\epsilon > 0$, the limit $\lim_{t\to\infty} u$ exists.

Theorem. Suppose that for some constant (P), the condition holds. Then, for any p satisfying the condition Re $p > \delta$, the solution u of the problem (14) stabilizes in the metric to the solution w of the corresponding stationary problem (10), (11).

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Note: Figure translations are in progress. See original paper for figures.