A GENERALIZATION OF A THEOREM OF M. INAGAKI

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Abstract

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CYBERNETICS AND THE THEORY OF REGULATION

I. G. GLOBENKO

A GENERALIZATION OF A THEOREM OF M. INAGAKI

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In the study of economic phenomena, substantial assistance may be provided by the analysis of mathematical models that make it possible, under simplified conditions, to study the phenomenon under consideration. Some mathematical models describing the process of growth of fixed assets and output in a one-product dynamic model under an optimal plan were indicated in [1]. The investigation of one-product models has attracted especially great attention in recent years. The purpose of the present note is to obtain an exact solution of the optimal growth trajectory of a certain one-product dynamic model.

A. We introduce the necessary definitions and notation. x denotes assets per unit of labor; y denotes the product produced, and c consumption per unit of labor per unit of time; L is the demographic function–the labor resources. The following assumptions are made:

$$y = x^{\alpha}, \quad 0 \le x < +\infty, \quad 0 < \alpha < 1;$$

$$L = L_0 e^{\lambda t}, \quad \lambda > 0, \quad L_0 > 0, \quad 0 \le t < \infty.$$

We have

$$c = y - (\lambda x + \dot{x}).$$

We further assume that the utility function has the form

$$u(c) = c^{1-\nu}/(1-\nu), \quad 0 < \nu < 1, \quad 0 < c < +\infty.$$

Definition 1. We shall call a function $x(t)$ an **admissible growth trajectory** if: 1) $x(0) = x_0,\ x_0 > 0$; 2) $x(t)$ is positive and piecewise continuously differentiable for $0 \le t \le \infty$; 3) $-\infty < \dot{x}(t) < x^{\alpha} - \lambda x,\ 0 \le t < \infty$, at all points of continuity of $\dot{x}(t)$.

Let $\{x(t)_0^\infty\}$ denote the set of admissible growth trajectories.

Definition 2. A growth trajectory $x^*(t) \in \{x(t)_0^\infty\}$ is **preferable** to a trajectory $x(t) \in \{x(t)_0^\infty\}$ if the condition

$$\lim_{T\to\infty} \{I[x^*(t)_0^T] - I[x(t)_0^T]\} \geq 0, \qquad I[x(t)_0^T] = \int_0^T u(c)\,dt$$

is satisfied.

Definition 3. We shall call an admissible growth trajectory $x(t)$ an **optimal trajectory** of the set $\{x(t)_0^\infty\}$ if it is preferable to any other trajectory from this set.

B. Let us denote the optimal growth trajectory by $\hat{x}(t)$. The corresponding y, c will be denoted by $\hat{y}(t), \hat{c}(t)$. We formulate our theorem.

Theorem. Let $\alpha = \nu$; $c_0, \bar{c}$ be the initial and asymptotic values of c; $\bar{c}$ is calculated by the Golden Rule, formulated in [2], and is equal to

$$\bar{c} = \nu^{\nu/(1-\nu)}(1-\nu)\lambda^{-\nu/(1-\nu)}. \tag{1}$$

Under these conditions the optimal trajectories $\hat{x}, \hat{y}, \hat{c}$ have the form

$$\begin{aligned} \hat{x} &= \left(\frac{\bar{c}}{1-\nu}\right)^{\frac{1}{\nu}} \left[1 - me^{-\frac{1-\nu}{\nu}\lambda t}\right]^{\frac{1}{1-\nu}}, \\ \hat{y} &= \frac{\bar{c}}{1-\nu} \left[1 - me^{-\frac{1-\nu}{\nu}\lambda t}\right]^{\frac{\nu}{1-\nu}}, \\ \hat{c} &= \bar{c} \left[1 - me^{-\frac{1-\nu}{\nu}\lambda t}\right]^{\frac{1}{1-\nu}}, \end{aligned} \tag{2}$$

where $m = 1 - [c_0/\bar{c}]^{1-\nu}$.

For $\nu = \alpha = \frac{1}{2}$ we obtain Inagaki' s theorem [3].

The proof of the theorem is based on Koopmans' theorem [4], according to which, under the assumptions of our theorem, in order that the growth trajectory $x(t)$ be optimal, it is necessary and sufficient that the condition

$$\dot{x}(t) = \frac{u(\bar{c}) - u(c)}{u'(c)}, \qquad 0 \leqslant t < +\infty. \tag{3}$$

be satisfied.

Equation (3), under our assumptions, has the form

$$\dot{x} = \frac{1}{1-\nu}\bar{c}^{\,1-\nu}c^{\nu} - \frac{1}{1-\nu}c. \tag{4}$$

Using $\dot{x} = -c + x^{\nu} - \lambda x$, we obtain

$$\frac{\nu}{1-\nu}c - \frac{\bar{c}^{\,1-\nu}}{1-\nu}c^{\nu} = \lambda x - x^{\nu}. \tag{5}$$

Differentiating (5), we obtain

$$\frac{\nu}{1-\nu}\dot{c} - \frac{\nu}{1-\nu}\bar{c}^{\,1-\nu}c^{\nu-1}\dot{c} = \lambda\dot{x} - \nu x^{\nu-1}\dot{x}. \tag{6}$$

Using (6) and (4), we find x and y:

$$x = \left[\dot{c}/c + \lambda/\nu\right]^{1/(\nu-1)}, \qquad y = \left[\dot{c}/c + \lambda/\nu\right]^{\nu/(\nu-1)}. \tag{7}$$

From equation (5) we find

$$x = (1-\nu)^{-1/\nu}\bar{c}^{(1-\nu)/\nu}c. \tag{8}$$

Equation (4), with the use of (8), takes the form

$$(1-\nu)^{(\nu-1)/\nu}\bar{c}^{(1-\nu)/\nu}\dot{c} = \bar{c}^{\,1-\nu}c^{\nu} - c.$$

Next,

$$dc/(\bar{c}^{\,1-\nu}c^{\nu} - c) = (1-\nu)^{(1-\nu)/\nu}\bar{c}^{(\nu-1)/\nu}dt,$$

$$\int_{c_0}^{c}\frac{d\xi}{\bar{c}^{\,1-\nu}\xi^{\nu} - \xi} = (1-\nu)^{(1-\nu)/\nu}\bar{c}^{(\nu-1)/\nu}t, \qquad \int_{c_0}^{c}\frac{d(\xi/\bar{c})}{(\xi/\bar{c})^{\nu} - \xi/c} = (1-\nu)^{(1-\nu)/\nu}\bar{c}^{(\nu-1)/\nu}t.$$

Put $\xi/\bar{c} = u$,

$$\int_{c_0/\bar{c}}^{c/\bar{c}}\frac{du}{u^{\nu} - u} = (1-\nu)^{(1-\nu)/\nu}\bar{c}^{(\nu-1)/\nu}t,$$

$$\frac{1}{1-\nu}\int_{c_0/\bar{c}}^{c/\bar{c}}\frac{du^{1-\nu}}{1 - u^{1-\nu}} = (1-\nu)^{(1-\nu)/\nu}\bar{c}^{(\nu-1)/\nu}t,$$

$$-\frac{1}{1-\nu}\ln\frac{1-(c/\bar{c})^{1-\nu}}{1-(c_0/\bar{c})^{1-\nu}} = (1-\nu)^{(1-\nu)/\nu}\bar{c}^{(\nu-1)/\nu}t, \tag{9}$$

whence, using (1), (7), and (9), we obtain (2). Let us note that the equation of optimal trajectories can be integrated in closed form also in the case of allowance for technical progress in the form of an exponential.

Institute of Mathematics
Siberian Branch of the Academy of Sciences of the USSR

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[4] T. C. Koopmans, Pontificiae Acad. sci. scripta varia, 28, 1965, p. 225–300.

Note: Figure translations are in progress. See original paper for figures.