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Abstract

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MATHEMATICS

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ON AN EXTENSION OF THE CLASS OF STABLE DISTRIBUTIONS

(Presented by Academician Yu. V. Linnik on 14 VI 1966)

Let $\xi_1, \xi_2, \ldots$ be a sequence of mutually independent random variables and

$$\zeta_n = \frac{1}{B_n} \sum_{i=1}^{n} \xi_i - A_n, \qquad n = 1, 2, \ldots, \tag{1}$$

a sequence of normalized sums which, for a suitable choice of the normalizing constants ($B_n \to \infty$), has its own limiting distribution $G(x)$. In the case when the summands ξ_n ($n = 1, 2, \ldots$) are identically distributed, the limiting law $G(x)$ is known (see ([1]), § 33) to be stable. B. V. Gnedenko proposed the problem of characterizing the class of limiting distributions $\{G(x)\}$ in the case when the condition of identical distribution of the summands is replaced by the weaker condition that the distributions of the summands belong to one type (to a finite number of types). More precisely, in the case of one, for example, type, one considers the class of limiting distributions for (1) under the condition that $F_{\xi_n}(x)$—the distribution function of the variable ξ_n—has the form

$$F_{\xi_n}(x) = F(a_n + b_{nx}), \qquad n = 1, 2, \ldots, \tag{2}$$

where a_n, $b_n > 0$ are certain real numbers, and $F(x)$ is a distribution function. The following restriction is imposed on the sequence $\{b_n,\ n = 1, 2, \ldots\}$:

$$0 < b' \leq b_n \leq b'' < \infty, \qquad n = 1, 2, \ldots \tag{3}$$

Denote by $\mathfrak{G}_1$ the class of proper limiting distributions for (1) when (2) and (3) are satisfied. It follows directly from the definition of $\mathfrak{G}_1$ that it includes all stable distributions ($b_n = \text{const}$, $n = 1, 2, \ldots$).

A description of the class $\mathfrak{G}_1$ in terms of the spectral functions of the representation of the logarithms of characteristic functions of limiting laws by P. Lévy' s formula (see ([1]), § 18) can be given by means of the following theorem.

Theorem. In order that a spectral function $H(x)$, not identically constant on at least one of the half-axes of definition, correspond, by P. Lévy' s formula, to a distribution law $G(x) \in \mathfrak{G}_1$, it is necessary and sufficient that on the half-axes of definition it admit the representation

$$H(\pm y) = \mp \int_{b'}^{b''} \Phi^{\pm}(z\beta)\, d_\beta \Psi\left(\beta, \frac{y}{z}\right), \qquad y > 0,\ z > 0, \tag{4}$$

where $\Phi^{\pm}(y)$ are nonnegative nondecreasing functions; $\Phi^{\pm}(\infty) = 0$; the function $\Psi(\beta, y)$ is, for each y, nonnegative and nondecreasing, and, moreover, if $y_1 < y_2$, then $\Psi(\beta, y_1) - \Psi(\beta, y_2)$ is also nonnegative and nondecreasing.

In this case the expansion holds

$$\int_0^y H(\pm y)\, \frac{dy}{y} = \mp y^{-\lambda} \left\{ c_0^{\pm} + \sum_{\mu=1}^{\infty} c_\mu^{\pm} \cos\left[\omega_\mu \log y + v_\mu^{\pm}\right] \right\}. \tag{5}$$

Here $0 < \lambda < 2$; $c_0^{\pm} \geq 0$, $c_0^{+} + c_0^{-} > 0$; $\sum_{\mu=1}^{\infty} |c_\mu^{\pm}| < \infty$, and the exponents λ, $\lambda \pm i\omega_\mu$ $(\mu = 1, 2, ...)$ are roots of a certain entire function of finite degree $\sigma(z)$ of the form

$$\sigma(z) = \int_{\beta'}^{\beta''} e^{z\beta}\, d\omega(\beta), \tag{6}$$

where $\omega(\beta)$ is of bounded variation.

Distribution laws with spectral functions of the form (5) constitute a special case of laws of a more general form, introduced by Yu. V. Linnik in paper (4) in connection with the study of identically distributed linear statistics in repeated samples.

In the expansion (5), stable distributions correspond to the case $c_\mu^{\pm} = 0$, $\mu = 1, 2, ...$ In terms of the representation (4), in order that the limiting distribution be stable, it is necessary and sufficient that the function $\Psi(\beta, y)$ have the form

$$\Psi(\beta, y) = \begin{cases} 0, & \beta < \bar{\beta}, \\ \psi(y), & \beta > \bar{\beta}. \end{cases} \tag{7}$$

The theorem stated above can also be extended to the case of several types. Denote by $\mathfrak{G}_r$ $(r = 1, 2, ...)$ the class of proper limiting distributions for (1), under the condition that

$$F^{\xi_n}(x) = F^{\eta_{j_n}}(a_n + b_n x), \qquad n = 1, 2, ... \tag{8}$$

Here the index j_n takes one of the values $1, 2, \ldots, r$; $F_j(x)$, $j = 1, 2, \ldots, r$, are distribution functions, and (3) holds. Each law belonging to the class $\mathfrak{G}_r$ can be represented as the composition of no more than r laws having spectral functions of the form (5).* In this case, a necessary and sufficient condition for a limiting law to belong to the class $\mathfrak{G}_r$, in terms of spectral functions, is the existence, for the spectral function $H_k(x)$ $(k = 1, 2, \ldots, \rho \leq r)$ of each of the components of the representation,

$$H_k(\pm y) = \sum_{l=1}^{l_k} \int_{b'}^{b''} \Phi_{kl}^{\pm}(z\beta)\, d\Psi_{kl}^{\pm}\left(\beta, \frac{y}{z}\right), \qquad y > 0,\ z > 0, \tag{9}$$

where $\Phi_{kl}^{\pm}(y)$ and $\Psi_{kl}(\beta, y)$ have the same nature as in (4). The special case corresponding to $b_n = \text{const}$, $n = 1, 2, \ldots$, was considered earlier by the author in [2,3].

In conclusion the author expresses gratitude to B. V. Gnedenko for posing the present problem.

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CITED LITERATURE

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2. A. A. Zinger, DAN, 162, No. 6, 1238 (1965).

3. A. A. Zinger, *Theory of Probability and Its Applications*, 10, 4, 672 (1965).

4. Yu. V. Linnik, *Ukrainian Mathematical Journal*, 5, 2, 207 (1953); 3, 247 (1953).

* A normal law may also be included in this number.

Note: Figure translations are in progress. See original paper for figures.