ALGEBRAIC THEORY OF LINEAR INEQUALITIES

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Abstract

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ALGEBRAIC THEORY OF LINEAR INEQUALITIES

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Finite systems of linear inequalities are usually studied under the assumption that the ground field is the field R of real numbers. This accounts for the possibility of using in their study not only the methods of linear algebra, supplemented of course by finitary methods of the theory of ordered fields (for example, the selection of the greatest element among a finite number of elements of an ordered field), but also infinitary methods connected with closedness and compactness and, in particular, methods based on the theorem on separation of closed convex sets in the space R^n. The methods of linear algebra with the addition just indicated will below be called simply **finitary methods**.

Recently there has been a tendency to study finite systems of linear inequalities under the assumption that the ground field is an arbitrary ordered field (see, for example, ([1,2])). Individual remarks on the validity of certain theorems of the theory of linear inequalities under this assumption already occur in the collection ([3]). The author of the present article has attempted, using only finitary methods, to construct a theory of finite systems of linear inequalities under so general an assumption on the ground field. For the sake of brevity we shall call it the **algebraic theory of linear inequalities** (a.t.l.i.).

In constructing the a.t.l.i. it turned out that many known theorems concerning linear inequalities can be proved by finitary methods (are finitarily conditioned) and hence can be included in the a.t.l.i. (and in this sense are algebraic results of the theory of linear inequalities). In constructing the a.t.l.i. two key results (see Theorems 1.1 and 1.3) of this theory were singled out, with the help of which all its results noted below can be obtained by finitary methods.

1. Here and everywhere below P will denote any ordered field, and $L(P)$ an arbitrary space over it. A system

$$f_j(x) - a_j \leq 0 \qquad (j = 1, 2, \ldots, m), \tag{1}$$

where $f_j(x)$ are linear (i.e., additive and homogeneous) functions on $L(P)$ with values in the field P, and the constant terms a_j are elements of P, will be called a system of linear inequalities over the space $L(P)$. In the special case where $L(P)$ is the space P^n of n-dimensional vectors over the field P, the system (1) has the form

$$f_j(x) - a_j = a_{j1}x_1 + \dots + a_{jn}x_n - a_j \leq 0 \qquad (j = 1, 2, \dots, m), \tag{2}$$

where all coefficients and constant terms are elements of P.

The system (1) is called **consistent** if there exists an element $x^0 \in L(P)$ satisfying all its inequalities, and **inconsistent** otherwise. The maximal number of linearly independent (over P) functions among the functions $f_j(x)$ is called the **rank** of the system (1). A solution of the system (1) of rank $r > 0$ is called its **nodal solution** if it turns into equalities some r of its inequalities with linearly independent (over P) functions.

$f_j(x)$. Two nodal solutions are not considered essentially different if they turn the same inequalities of system (1) into equalities. Some subsystem of r inequalities of system (1) of rank $r > 0$ will be called a **nodal subsystem** if its rank is equal to r and if all its nodal solutions satisfy system (1).

Theorem 1.1. *Every consistent system (1) of nonzero rank has at least one nodal subsystem, and hence at least one nodal solution.*

A linear inequality $f(x) - a \leqslant 0$ (over $L(P)$) is called a **consequence** of the consistent system (1) if all its solutions satisfy it.

Theorem 1.2. *If the linear inequality $f(x) - a \leqslant 0$ is a consequence of the consistent system (1) of nonzero rank, then there exists an element $a^* \in P$ and a nodal subsystem of system (1) such that all its solutions satisfy the inequality $f(x) - a^* \leqslant 0$, and all its nodal solutions satisfy the equation $f(x) - a^* = 0$.*

Theorem 1.3. *If the linear inequality $f(x) - a \leqslant 0$ is a consequence of the consistent system (1), then there exist nonnegative elements $p_0, p_1, \dots, p_m$ of the field P for which the following relation holds identically with respect to $x \in L(P)$:*

$$f(x) - a = \sum_{j=1}^{m} p_j(f_j(x) - a_j) - p_0.$$

Theorem 1.2 can be derived by finitary methods from Theorems 1.1 and 1.3. It follows directly from Theorem 1.2 that if a linear function $f(x)$, $(x \in L(P))$, is bounded above on the set M of solutions of a consistent system (1) of nonzero rank, then among its values on the set M there exists a greatest value. This proposition can be derived (by finitary methods) from Theorem 1.1.

For the case when the field P is the field R of real numbers, the propositions given here are known; in particular, Theorem 1.3 is known under the name of the Minkowski–Farkas theorem; Theorems 1.1 and 1.2 are contained in previously published works of the author.

2. For the case of system (2), Theorem 1.1 implies the validity of the following proposition.

Theorem 2.1. *A necessary and sufficient condition for the consistency of a system of linear inequalities of rank $r > 0$ with coefficients and constant terms from an arbitrary ordered field P is the existence in the matrix of its coefficients of a nonzero minor*

$$\Delta = \begin{vmatrix} a_{j_1 i_1} & \cdots & a_{j_1 i_r} \\ \cdots & \cdots & \cdots \\ a_{j_r i_1} & \cdots & a_{j_r i_r} \end{vmatrix}$$

of order r, such that the relations

$$\frac{1}{\Delta} \begin{vmatrix} a_{j_1 i_1} & \cdots & a_{j_1 i_r} & a_{j_1} \\ \cdots & \cdots & \cdots & \cdots \\ a_{j_r i_1} & \cdots & a_{j_r i_r} & a_{j_r} \\ a_{j i_1} & \cdots & a_{j i_r} & a_j \end{vmatrix} \geqslant 0 \qquad (j = 1, 2, \dots, m).$$

For $P = R$ this proposition was established in article (4) and is its key proposition. With the help of Theorem 2.1 and Theorem 1.3 it can be shown that all the results of article (4) are finitely determined and, consequently, that they can be included in a.t.l.n.

Using Theorem 1.1, it is not difficult to prove the following proposition:

Theorem 2.2. *If some element of the arbitrary space $L(P)$ is linearly expressed with nonnegative coefficients from P through some system of elements of $L(P)$ having nonzero rank, then it is linearly expressed with nonnegative coefficients (from P) through some linearly independent subsystem of it.*

Using Theorem 1.3 and Theorems 2.1 and 2.2, one can verify that all the results of the article [5] can be included in the a.t.l.i.

A number of other algebraic results concerning linear inequalities are contained in the articles [1, 2] (their algebraic character is in fact also established in these articles). It turned out that all of them can be derived by finitary methods from Theorems 1.1 and 1.3.

Using Theorems 1.3 and 2.2, it is not difficult to verify the finitary conditionality of all the results of the article by E. Burger [6], which gives a method for obtaining the general solution of a system of homogeneous linear inequalities, and also the finitary conditionality of all the results of the article by S. N. Chernikov [7], which gives a method for reducing systems of linear inequalities. Therefore these methods can also be included in the a.t.l.i. Using Theorems 1.2 and 1.3, it is not difficult to verify that all the basic theorems of the theory of linear programming can likewise be included in the a.t.l.i.

Let us also note that, using Theorems 1.2 and 1.3, the following proposition can be obtained by finitary methods.

Let $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_n)$ be two arbitrary elements of the set S of nonnegative elements of the space P^n (P is any ordered field) with the sum of their coordinates equal to one, and let (a_{ij}) be any matrix of degree n. Then there exist and are equal to one another

$$\min_{y \in S} \max_{x \in S} \sum_{i,j=1}^{n} a_{ij} x_i y_j, \qquad \max_{x \in S} \min_{y \in S} \sum_{i,j=1}^{n} a_{ij} x_i y_j.$$

For $P = R$ this proposition is known under the name of von Neumann's minimax theorem (see [8]).

3. A set in $L(P)$ that decomposes into a direct sum of a finitely generated convex cone over P and a subspace having a finite-dimensional direct complement in $L(P)$ will be called a **finitely reducible** cone of the space $L(P)$.

Theorem 3.1. *The set M of solutions of a consistent system (1) of rank different from zero is the sum (in the algebraic sense) of a finitely reducible cone C of solutions of the corresponding system $f_j(x) \leqslant 0$ $(j = 1, 2, \dots, m)$ of homogeneous linear inequalities and a centroid E, generated (over P) by the set of essentially distinct nodal solutions of the system (1). If some set F of the space $L(P)$ decomposes into the sum (in the algebraic sense) of a finitely reducible cone A and a finitely generated centroid H, then it is the set of solutions of at least one system of linear inequalities over $L(P)$ (i.e. of the form (1)); moreover, for any of them the set of solutions of the corresponding system of homogeneous inequalities coincides with the cone A.*

For a system of the form (2) over the space R^n this proposition is known.

Theorem 3.1 is obtained by finitary methods with the aid of the following propositions.

1) The set of solutions of the system $a_{j1}x_1 + \dots + a_{jn}x_n \leqslant 0$ $(j = 1, 2, \dots, m)$ (over P^n) is a finitely generated (convex) cone (over P).

2) If $(b_{k1}, \dots, b_{kn})$ $(k = 1, 2, \dots, l)$ is some system of generating elements of this cone, then the set of solutions of the system $b_{k1}x_1 + \dots + b_{kn}x_n \leqslant 0$ $(k = 1, 2, \dots, l)$ coincides with the (convex) cone generated by the elements $(a_{j1}, \dots, a_{jn})$ $(j = 1, 2, \dots, m)$ (over P).

For $P = R$ these propositions are known and constitute the so-called **duality principle** of the theory of linear inequalities. Proposition 2) follows directly from Proposition 1) and Theorem 1.3. Proposition 1) is proved elementarily by means of finitary methods (see, for example, [1]):

4. Let U be some subspace of $L(P)$. For system (1) (over $L(P)$) compose the equation $u_1 f_1(x) + \dots + u_m f_m(x) = 0$ $(x \in U)$ with unknowns $u_1, \dots, u_m$. Its nonnegative solution $(u_1^0, \dots, u_m^0)$ $(u_1^0, \dots$

$\dots, u_m^0 \in P)$ shall be called a **fundamental solution** if the rank of the system of functions $f_j(x)$ corresponding to the nonzero u_j^0 is one less than their number. If $(u_1^k, \dots, u_m^k)$ $(k = 1, 2, \dots, l)$ is a maximal system of its fundamental solutions, differing from one another in the number of at least one nonzero coordinate, then the system

$$\sum_{j=1}^{m} u_j^k f_j(x) - \sum_{j=1}^{m} u_j^k a_j \leqslant 0 \qquad (k = 1, 2, \dots, l)$$

over $L(P)$ will be called the **fundamental U-convolution** of system (1). If it has no nonzero nonnegative solutions, then we shall say that the fundamental U-convolution of system (1) is empty and that it is satisfied by all elements of $L(P)$. Two consistent systems of linear inequalities over the space $L(P)$ will be called **U-homomorphically equivalent** if the sets of their solutions coincide modulo U.

Theorem 4.1. *If system (1) is consistent, then its fundamental U-convolution with respect to any subspace U of $L(P)$ is consistent or empty. If in $L(P)$ there exists such a subspace U with respect to which the fundamental U-convolution of system (1) is consistent or empty, then system (1) is consistent.*

Theorem 4.2. *Two consistent systems of linear inequalities over the space $L(P)$ are U-homomorphically equivalent if and only if their fundamental U-convolutions are equivalent (i.e., have the same solutions).*

It follows from Theorem 4.2 that if two consistent systems of the form (1) are U-homomorphically equivalent, then the projections of the sets of their solutions onto any direct complement V of the subspace U in $L(P)$ coincide; and conversely, if for some subspace U of $L(P)$ there exists a direct complement V in $L(P)$ such that the projections onto it of the sets of their solutions coincide, then they are U-homomorphically equivalent. For $P = R$, Theorem 4.1 was published earlier by the author in paper ([9]) (see also ([7])). The question of homomorphic equivalence is considered here for the first time.

5. A **convex polyhedral set** of the space $L(P)$ will mean the set of solutions of an arbitrary consistent system of the form (1) over $L(P)$. We shall say that two sets A and B of the space $L(P)$ are **separated** by the plane $f(x) - a = 0$ of the space $L(P)$ ($f(x)$ is a nonzero linear function on $L(P)$, and $a \in P$), if $f(x) - a \leqslant 0$ for all elements of one of these sets and $f(x) - a \geqslant 0$ for all elements of the other. If the equality $f(x) - a = 0$ is not satisfied here for a single $x \in A$ nor for a single $x \in B$, then we shall say that the plane $f(x) - a = 0$ **strictly separates** the sets A and B. The sets A and B will be called **tangent** if they have common elements

and if there exists an element $c \in L(P)$ such that, under the displacement of one of them by the element ct with arbitrary $t > 0$ ($t \in P$), a set is obtained that has no common elements with the other.

Theorem 5.1. *If two convex polyhedral sets of the space $L(P)$ have no common elements, then there exists a plane of the space $L(P)$ strictly separating these sets.*

Theorem 5.2. *Any two tangent convex polyhedral sets of the space $L(P)$ are separated (non-strictly) by some plane from $L(P)$.*

For $P = R$ these theorems were formulated in the author' s paper ([10]).

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CITED LITERATURE

[1] A. Charnes, W. W. Cooper, Proc. Nat. Acad. Sci. U.S.A., **44**, No. 9, 914 (1958).
[2] A. Ben-Israel, J. Math. Analysis and Appl., **9**, No. 2, 303 (1964).
[3] Collection: *Linear Inequalities and Related Questions*, IL, 1959.
[4] S. N. Chernikov, UMN, **8**, 2 (54), 7 (1953).
[5] S. N. Chernikov, Matem. sborn., **38** (80), 4, 479 (1956).
[6] E. Burger, Zs. angew. Math. u. Mech., **36**, No. 3/4, 135 (1956).
[7] S. N. Chernikov, Zhurn. vychisl. matem. i matem. fiz., **5**, No. 1, 3 (1965).
[8] E. F. Beckenbach, R. Bellman, *Inequalities*, 1961.
[9] S. N. Chernikov, DAN, **152**, No. 5, 1075 (1963).
[10] S. N. Chernikov, DAN, **138**, No. 6, 1298 (1961).

Note: Figure translations are in progress. See original paper for figures.