UNITARY REPRESENTATIONS OF THE GROUP $GL(3,R)$ OF REAL NONSINGULAR MATRICES OF THIRD ORDER

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Abstract

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UNITARY REPRESENTATIONS OF THE GROUP $GL(3, R)$ OF REAL NONSINGULAR MATRICES OF THIRD ORDER

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In this note all irreducible unitary representations of the group $GL(3, R)$ are listed.

Consider the subgroup $K \subset GL(3, R)$ of matrices $\|a_{ij}\|$ $(i = 1, 2, 3)$ with the condition $a_{31} = a_{32} = 0$. For a matrix $k \in K$, denote by $A(k)$ the diagonal block of size 2×2 situated in the left upper corner. Suppose that a representation of the subgroup K has the form

$$k \to |a_{33}|^{i\rho} \operatorname{sgn}^{\varepsilon}(a_{33}) T_{A(k)}, \tag{1}$$

where $k \in K$; ρ is a real number; the index $\varepsilon = 0, 1$; $T_{A(k)}$ is any irreducible unitary representation of the group $GL(3, R)$.

Theorem 1. *Every irreducible unitary representation of the group $GL(3, R)$ is either one-dimensional or equivalent to a representation induced from the stationary subgroup K and an inducing representation of the form* (1).

For the classification of all representations of the group $GL(3, R)$, it remains to indicate which of the representations appearing in Theorem 1 are equivalent. We first list all irreducible unitary representations of the group $GL(2, R)$. This is easy to do, knowing all representations of the unimodular group $SL(2, R)$, found by Bargmann ([1]). It turns out that there exist 4 series of irreducible unitary representations of the group $GL(2, R)$.

I. The continuous series is realized in $L^2(-\infty, \infty)$:

$$T_{\begin{pmatrix} a & b \\ c & d \end{pmatrix}} f(x) = f\left(\frac{ax + c}{bx + d}\right) \left|\frac{bx + d}{ad - bc}\right|^{-1+i\rho_1} \operatorname{sgn}^{\varepsilon_1}\left(\frac{bx + d}{ad - bc}\right) |ad-bc|^{i\rho_2} \operatorname{sgn}^{\varepsilon_2}(ad-bc).$$

II. The discrete series is realized in the space $\mathcal{H}_s$ ($s = 1, 2, 3, ...$) of functions analytic in the upper and lower half-planes separately, with the condition $\varphi \in \mathcal{H}_s$ if

$$\int |\varphi(z)|^2 |\operatorname{Im} z|^{s-1}\, dz\, d\bar{z} < \infty,$$

$$T_{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)} f(z) = f\left(\frac{az+c}{bz+d}\right)\left(\frac{bz+d}{ad-bc}\right)^{-s-1} |ad-bc|^{i\rho}.$$

III. The supplementary series is realized in the space L_s ($0 < s < 1$) of functions on the line with scalar product defined by the kernel $|x_1 - x_2|^{s-1}$:

$$T_{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)} f(x) = f\left(\frac{ax+c}{bx+d}\right)\left(\frac{bx+d}{ad-bc}\right)^{-s-1} |ad-bc|^{i\rho} \operatorname{sgn}^{\varepsilon}(ad-bc).$$

IV. The degenerate series consists of one-dimensional representations:

$$T_{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)} = |ad-bc|^{i\rho} \operatorname{sgn}^{\varepsilon}(ad-bc).$$

Representations of the group $GL(3, R)$ obtained from representations of types I–IV will be called, respectively, the **continuous**, **discrete**, **supplementary**, and **degenerate** series.

In the case of the continuous series, the representation of the subgroup K itself is induced by a unitary representation of the subgroup $K_0 \subset K$, consisting of upper triangular matrices. For this reason the representation of the continuous series of the group $GL(3, R)$ is also induced by a unitary representation of K_0. The inducing representation will be one-dimensional, of the form

$$k_0 \to \prod_{i=1}^{3} |a_{ii}|^{\rho_i} \operatorname{sgn}^{\varepsilon_i}(a_{ii}), \qquad k_0 \in K_0. \tag{2}$$

Theorem 2. *Among the induced representations of the group $GL(3, R)$ listed above, equivalence occurs only among representations of the continuous series. Let two representations of the subgroup K_0 be given by the sets $(\rho_i)(\varepsilon_i)$ and $(\rho_i')(\varepsilon_i')$, respectively ($i = 1, 2, 3$; see (2)). Then the representations of the group $GL(3, R)$ induced by them are equivalent if and only if the sets (ρ_i') and (ε_i') are obtained from the sets (ρ_i) and (ε_i), respectively, by one and the same permutation.*

The method we use for finding all representations is based on a theorem of A. A. Kirillov ([2]), according to which the restriction of an irreducible unitary representation of the group $GL(3, R)$ to the subgroup K remains irreducible. All representations of the subgroup K and the corresponding representations of the Lie algebra K are considered. From the commutation relations one finds the Lie operators of the missing one-parameter subgroups. The main difficulty here lies in the fact that, a priori, the Gårding space or any other natural domain of definition of the Lie operators for the group $GL(3, R)$ is not known in advance. One device that makes it possible to overcome this difficulty consists in studying, among the missing Lie operators, the one that corresponds to a compact one-parameter subgroup. For this operator it turned out to be possible to find all self-adjoint extensions to which a representation of $GL(3, R)$ may correspond.

We note that only the continuous and discrete series enter the Plancherel formula found by B. D. Romm ([3]). (In his work these series are given for the unimodular group and are denoted by d_0 and d_1, respectively; moreover, for the cases d_1 with $\varepsilon = 0$ and $\varepsilon = 1$ (formula (1) of ([3])) the representations are equivalent.) These basic series were first indicated by I. M. Gelfand and M. I. Graev in ([4]).

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CITED LITERATURE

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3. B. D. Romm, DAN, **160**, No. 6, 1269 (1965).
4. I. M. Gelfand, M. I. Graev, Izv. AN SSSR, **17**, No. 3, 189 (1953).

Note: Figure translations are in progress. See original paper for figures.