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MATHEMATICAL PHYSICS

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**Abstract**

**Full Text**

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*MATHEMATICAL PHYSICS*

P. E. KRASNUSHKIN

**THE USE OF EVOLUTION GRAPHS OF FUNCTIONAL NETWORKS IN SOLVING BOUNDARY-VALUE PROBLEMS OF ELECTRODYNAMICS BY THE METHOD OF MATCHING**

*(Presented by Academician I. M. Vinogradov, June 17, 1965)*

1. Let a volume  $V$ , in which Maxwell's equations for the complex amplitudes  $E$  and  $H$  hold, be divided into partial regions  $i = 1, 2, 3, \dots, P$ . The solution of the first boundary-value problem of electrodynamics <sup>(1,3)</sup> for a region  $i$  adjacent to other regions  $i'$  along surfaces  $S_j^{(i)}$ ,  $j = 1, 2, 3, \dots, M^{(i)}$ , establishes a unique functional relation between the tangential components of the fields  $E_k^{(i)}$  and  $H_j^{(i)}$  on  $S_j^{(i)}$ :

$$\{H_j^{(i)}\}_{j=1}^M = (Y_{jk}^{(i)}) \{E_k^{(i)}\}_{k=1}^M, \quad (1)$$

where  $(Y_{jk}^{(i)})$  is an  $M \times M$  matrix composed of the linear conductivity operators <sup>(1)</sup>. The continuity equations for  $E_k$  and  $H_j$  on  $S_j^{(i)}$  will be

$$E_{k_m}^{(i'')} = \mathcal{E}H_{j_m}^{(i')}, \quad m = 1, 2, \dots, R, \quad (2)$$

where  $\mathcal{E}$  is the identity-transformation operator. The set of partial regions is a functional network of  $M \times M$  cells ( $M$  input and  $M$  output channels), connected by ideal links described by the system of operator equations (1), (2). The solution of the first boundary-value problem for the volume  $V$  is now reduced to the solution of the network equations. Having represented (1), (2) by a two-color evolution graph according to the rules given in <sup>(2)</sup>, we carry out the investigation and solution of (1), (2) graphically.

2. Consider the case of a partition (Fig. 1a) in which the network is a chain of adjacently interacting  $2 \times 2$  cells, described by  $2 \times 2$  matrices composed of conductivity operators  $Y_{11}^{(i)}$ ,  $Y_{12}^{(i)}$ ,  $Y_{21}^{(i)}$ , and  $Y_{22}^{(i)}$ ,  $i = 1, 2, 3, \dots, R-1$ . At the ends of the network two-channel cells  $i = 0$  and  $i = R$  are connected,

with input-conductivity operators  $Y^{(0)} = [Z^0]^{-1}$  and  $Y^{(R)} = [Z^{(R)}]^{-1}$ . A time-harmonic ( $e^{-i\omega t}$ ) external action is localized in the section  $S_{i+1}$  and is specified by the amplitude functions  $E_0$  and  $H_0$ . The evolution graph of such a network is shown in Fig. 1b. For its construction an inversion of the arcs  $Y_{11}^{(i)}$ ,  $PR - (P = 2, R = 2)$  of the graphs of all cells,  $i = 1, 2, \dots, R-1$ , has been carried out according to the rules <sup>(2)</sup>. The right half of the graph in Fig. 1b has one path connecting the vertices  $4^{(i+1)}$  with  $1^{(i+1)}$  (42142 ... 1421 - 431431431 ... 43141), transmitting states from left to right and then back. The left half of the graph in Fig. 1b also has one path connecting the vertices  $3^{(i)}$  with  $2^{(i)}$  (314314 ... 431-421421 ... 4 ... 42), transmitting states from right to left and back. Both paths are loaded with feedback loops, and for computing the operators of the loaded paths <sup>(2)</sup> it is convenient to apply the recurrence formula (3), obtained directly from the graph of Fig. 1b, where  $\tilde{Y}_i$  is the path operator between the vertices  $1^{(i)}$  and  $4^{(i)}$ , and  $\tilde{Y}_{i+1}$  is that between  $1^{(i+1)}$  and  $4^{(i+1)}$ ,

$$\tilde{Y}_i = Y_{11}^{(i)} + Y_{21}^{(i)} [\tilde{Y}_{i+1} - Y_{22}^{(i)}]^{-1} Y_{21}^{(i)}. \quad (3)$$

Let us denote the obtained path operators between  $1^{(i+1)}$  and  $4^{(i+1)}$ , and between  $3^{(i)}$  and  $2^{(i)}$ , respectively, by  $\tilde{Y}_i^{(R)}$  and  $\tilde{Y}_i^{(0)}$  (these are  $Y^{(R)}$  and  $Y^{(0)}$ , recalculated from the ends of the chain in Fig. 1a to the section  $i, i + 1$ ). Now the graph in Fig. 1b can be replaced by the graph in Fig. 1c, and with its aid one can determine the  $2 \times 2$  matrices of the operators  $Z_1$  and  $Z_2$ , relating

$$\left\| \begin{array}{c} E_1^{(i+1)} \\ H_1^{(i+1)} \end{array} \right\| \quad \text{and} \quad \left\| \begin{array}{c} E_2^{(i)} \\ H_2^{(i)} \end{array} \right\| \quad \text{to} \quad \left\| \begin{array}{c} E_0 \\ H_0 \end{array} \right\|:$$

$$Z_1 = \begin{pmatrix} -\tilde{Z}_i^{(R)} \Pi'' \tilde{Y}_i^{(0)} & \tilde{Z}_i^{(R)} \Pi'' \\ \Pi'' \tilde{Y}_i^{(0)} & -\Pi'' \end{pmatrix}, \quad Z_2 = \begin{pmatrix} \Pi' & \Pi' \tilde{Z}_i^{(R)} \\ \tilde{Y}_i^{(0)} \Pi' & \tilde{Y}_i^{(0)} \Pi' \tilde{Z}_i^{(R)} \end{pmatrix}; \quad (4)$$

here  $\Pi'' = (\mathcal{E} + \tilde{Y}_i^{(0)} \tilde{Z}_i^{(R)})^{-1}$  and  $\Pi' = (\mathcal{E} + \tilde{Z}_i^{(R)} \tilde{Y}_i^{(0)})^{-1}$  are the feedback-loop operators of the circuit 6145326 with contour operators  $(-\tilde{Y}_i^{(0)} \tilde{Z}_i^{(R)})$  and  $(-\tilde{Z}_i^{(R)} \tilde{Y}_i^{(0)})$ , counted from vertices 5 for  $\Pi'$  and 6 for  $\Pi''$ . The graph in Fig. 1c gives the solution for forced oscillations in the volume  $V$ . They exist if the operator of the circuit 6145326 differs from  $\mathcal{E}$ , which follows from (2). Otherwise, natural oscillations exist. To find the fields  $E^{(i)}$  and  $H^{(i)}$  in any section  $i$ , let us replace, by inverting the arcs, the matrices of the conductance operators  $(Y_{jk}^{(i)})$  (the graph in Fig. 1b) by the Breisig matrices  $(A_{jk}^{(i)})$  (1), explicitly relating  $E_2^{(i)}$  and  $H_2^{(i)}$  to  $E_1^{(i)}$  and  $H_1^{(i)}$ .

**Fig. 1**

Fig. 1

Figure 1: Fig. 1

Combining the states  $E_1^{(i)}$  and  $H_1^{(i)}$  into one, determined by the vector-function

$$\left\| \begin{array}{c} E_1 \\ H_1 \end{array} \right\|,$$

we replace the graph in Fig. 1b by the simpler graph in Fig. 1d. With its help it is easy to construct expressions relating

$$\left\| \begin{array}{c} E_1^{(i')} \\ H_1^{(i')} \end{array} \right\|$$

to  $E_0^{(i+1)}$  and  $H_0^{(i+1)}$  in any section  $i', i' + 1$  (see (1)).

3. If the cells into which the volume  $V$  is divided are identical and are loaded at the ends by the cells 0 and  $R$  with  $Y^0$  and  $Y^R$ , determined from equation (3), where

$\tilde{Y}_i = \tilde{Y}_{i+1}$  (for the case  $Y_{11} = Y_{22}$  and  $Y_{12} = Y_{21}$ ,  $Y^{0,R} = Y_{11} \pm Y_{12}$ ), then it makes sense to carry out further investigation of the chain by the method of normal waves [4]. The problem of finding the normal waves can be reduced to the problem of the natural oscillations of a cell closed by a conditional waveguide according to the periodicity condition

$$E_2 = -e^{-i\psi} E_1, \quad H_2 = -e^{-i\psi} H_1.$$

As a result we obtain the closed graph of Fig. 2a. Reading from it the system of equations for  $E_1$ , we obtain

$$\begin{aligned} (Y_{11} + Y_{22})E_1 &= \\ &= (e^{i\psi}Y_{21} + e^{-i\psi}Y_{12})E_1, \end{aligned}$$

where  $Y_{ik}$  are the conductance operators of the cell.

4. Let us apply the stitching method to such a cell and divide it into partial regions (for example, according to Fig. 2b) with the prescribed matrices of the conductance operators (1). Joining the cells  $i = I$  and  $i = II$  according to the periodicity and continuity conditions, we obtain the graph of Fig. 2c. It consists of a single isolated contour. Computing the operator of this contour and equating it to  $\mathcal{E}$ , we obtain a system of homogeneous equations

$$\begin{pmatrix} Y_{11}^{(I)} - Y_{22}^{(II)} & Y_{12}^{(I)} + e^{i\psi} Y_{21}^{(II)} \\ Y_{21}^{(I)} + Y_{12}^{(II)} e^{-i\psi} & Y_{22}^{(I)} - Y_{11}^{(II)} \end{pmatrix} \times \begin{pmatrix} \|E_1\| \\ \|E_2\| \end{pmatrix} = 0 \quad (5)$$

$E_1$  and  $E_2$  are functions of the fields  $E_\tau$  in sections  $\langle 1 \rangle$  and  $\langle 2 \rangle$ . In Fig. 2c another division of the cell into partial regions  $I$  and  $II$  is given. Region  $I$  corresponds to a 6-channel, and  $II$  to a 2-channel, cell of the network described by the matrices (1). The graph of such a division is presented in Fig. 2d. It also consists of a single isolated contour. Equating the operator of this contour to  $\mathcal{E}$ , we obtain an equation for  $E_3$  in section  $\langle 1 \rangle$ :

$$\begin{aligned} & [(Y_{11} - Y_{22} - e^{-i\psi} Y_{12} + e^{i\psi} Y_{21})(Y_{31} - e^{-i\psi} Y_{32})^{-1}(Y_{00} - Y_{33}) + \\ & + (Y_{13} + e^{i\psi} Y_{33})] E_3 = 0. \end{aligned} \quad (6)$$

(5) and (6) are valid for regions  $I$  and  $II$  of arbitrary shape. The conditions of their compatibility are the dispersion equations relating the wave numbers of the normal waves  $\psi$  with the frequency  $\omega$ :  $\psi = f[\omega]$ . For numerical calculations they must be algebraized. If  $I$  and  $II$  are cylindrical waveguide sections with arbitrary contours  $\Gamma_I$  and  $\Gamma_{II}$  of the cross sections (Fig. 2e), then, introducing the coordinates  $r, \varphi, z$ , we represent  $(Y_{j,k}^{1,2})$  by diagonal matrices in the basis of normal waves traveling along the  $z$ -axis. In this case  $E$  and  $H$  are expanded in terms of the forms of the normal waves  $e_{m,n}^{(1)}(r, \varphi)$  and  $e_{m,n}^{(2)}(r, \varphi)$ , traveling along  $z$  in  $I$  and  $II$  with wave numbers  $\alpha_{m,n}$  and  $\gamma_{m,n}$ , respectively. If the fields  $E_1$  and  $E_2$  are approximated by Fourier series in the basis  $\{\mathcal{E}_s\}_s$ , different from  $\{e_{m,n}\}_{m,n}$ , i.e.

$$E_1 = \sum_{s=1}^N K_s^{(1)} \mathcal{E}_s \quad \text{and} \quad E_2 = \sum_{s=1}^N K_s^{(2)} \mathcal{E}_s(r, \varphi),$$

then the scalar pro-

admittances  $(Y_{j,k} \mathcal{E}_s, \mathcal{E}_{s'})$ , which are elements of the matrices  $\|Y_{jk}\|$ , will be convolutions over the indices  $m$  and  $n$  (see formula (8)). The dispersion equation of the waveguide composed of the cells of Fig. 2e, in matrix form, will be

$$\det \begin{bmatrix} a - \alpha - \beta \cos \psi & -\beta \sin \psi \\ -\beta \sin \psi & a + \alpha + \beta \cos \psi \end{bmatrix} = 0, \quad (7)$$

where  $a, \alpha$ , and  $\beta$  are square  $N \times N$  matrices with elements

$$a_{s,s} = \sum_{m,n} \frac{\text{cth} \gamma_{m,n} d}{\gamma_{m,n}} \beta_{m,n,s} \beta_{m,n,s'} + \sum_{m,n} \frac{\text{cth} \alpha_{m,n} \tau}{\alpha_{m,n}} \alpha_{m,n,s} \alpha_{m,n,s'}, \quad (8)$$

$$\alpha_{s,s'} = \sum_{m,n} \frac{\alpha_{m,n,s} \alpha_{m,n,s'}}{\alpha_{m,n} \operatorname{sh} \alpha_{m,n} \tau}, \quad \beta_{s,s'} = \sum_{m,n} \frac{\beta_{m,n,s} \beta_{m,n,s'}}{\gamma_{m,n} \operatorname{sh} \gamma_{m,n} d},$$

where  $\alpha_{m,n,s}$  and  $\beta_{m,n,s}$  are the Fourier coefficients of the expansion of  $\mathcal{E}_s$  in  $e_{m,n}^{(1)}$  and  $e_{m,n}^{(2)}$ , respectively. Below we consider cases in which branches  $I$  and  $II$  have circular cross sections (Fig. 2). Restricting ourselves to the case of axially symmetric waves of type TH, we obtain, for  $\alpha_{m,n,s} = \alpha_{m,s}$  and  $\beta_{m,n,s} = \beta_{m,s}$ , the following results.

If  $\{\mathcal{E}_s\}$  coincides with one of the bases of normal waves, for example with  $\{e_m(r/a)\}$ , then

$$\alpha_{m,s} = \delta_{m,s} = \begin{cases} 1, & m = s, \\ 0, & m \neq s; \end{cases} \quad \beta_{m,s} = \frac{2ab\rho_{0,m} J_0(\rho_{0,m} a/b) J_1(\rho_{0,s})}{J_1^2(\rho_{0,m}) (\rho_{0,s}^2 b^2 - \rho_{0,m}^2 a^2)}. \quad (9)$$

To take into account the electrostatic singularity of  $E$  at the sharp edges of the cell, we choose the basis

$$\mathcal{E}_s = (r/a) / [1 - (r/a)^2]^{1/p};$$

then

$$\beta_{m,s} = \frac{2\sqrt{\pi a^2}}{b |J_1(\rho_{0,m})|} \sum_{r=0}^{s-1} (-1)^r {}_{s-1}C_r^r 2^{r-1/p} \Gamma\left(r - \frac{1}{p} + 1\right) I_{r-1/p+2}\left(\rho_{0,m} \frac{a}{b}\right). \quad (10)$$

$$\alpha_{m,s} = \beta_{m,s} \quad (b = a).$$

For the case of subdividing the cell according to Fig. 2, when  $Y_{12} = Y_{21}$ ,  $Y_{23} = Y_{32}$ ,  $Y_{13} = Y_{31}$ ,  $Y_{11} = Y_{22}$ ,  $Y_{13} = Y_{23}$ , and equation (6) takes the form  $[Y_{13} Y_{12}^{-1} Y_{13} - Y_{33} + Y_0] E_3 = 0$ , it is convenient in region  $I$  to use as a basis the forms of normal waves traveling along  $r$ , with wave forms  $\exp\{-i\beta_m z\}$ ,  $\beta_m = (\psi + 2\pi m)/D$  (wave numbers  $\mu_m = \sqrt{k_0^2 - \beta_m^2}$ ), and for  $II$  the basis of the forms of normal waves  $\cos \pi p z/d$ , likewise traveling along  $r$ , with wave numbers  $\chi_p = \sqrt{k_0^2 - (p\pi/d)^2}$ . As a result of algebraization, we obtain the dispersion equation in the form

$$\det \left[ \sum_{m=-\infty}^{\infty} \frac{J_1(\mu_m a)}{\mu_m J_0(\mu_m a)} \beta_{m,s} \beta_{m,s'} - \sum_{p=0}^{\infty} \frac{D(a', b)}{\chi_p D(a, b)} \alpha_{p,s} \alpha_{p,s'} \right] = 0, \quad (11)$$

where  $D(a, b)$  and  $D(a', b)$  are given in (5).

V. A. Steklov Mathematical Institute  
of the Academy of Sciences of the USSR

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*Note: Figure translations are in progress. See original paper for figures.*

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