# APPROXIMATELY MINIMAX DETECTION OF A VECTOR SIGNAL ON A GAUSSIAN BACKGROUND

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## Abstract

## Full Text

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**MATHEMATICS**

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# APPROXIMATELY MINIMAX DETECTION OF A VECTOR SIGNAL ON A GAUSSIAN BACKGROUND

This note adjoins the note ([2]). A signal is considered in the form of a $p$-dimensional column vector $\xi$, and the background in the form of a normal vector $X - \xi \in N(0, \Sigma)$, where $\Sigma$ is an unknown nondegenerate correlation matrix. A repeated sample $X_1, \dots, X_N$ is taken. The hypothesis of absence of signal $H_0: \ \xi = 0$ is tested against the composite hypothesis $H_1: \ N\xi^T\Sigma^{-1}\xi = \delta$, and also against the composite hypothesis $H_1': \ N\xi^T\Sigma^{-1}\xi \geqslant \delta$, where $\delta > 0$ is a given number.

The expression $N\xi^T\Sigma^{-1}\xi$ may be interpreted as the relative energy of the signal, and the two problems posed—as problems of detecting a signal of prescribed relative energy $\delta$, or of relative energy not less than $\delta$. We seek a minimax procedure for signal detection, i.e., such a (randomized) test $\Phi$ for which, at a given level $\alpha$, the minimum power on the composite alternative $H_1$ (or $H_1'$) is maximal among all tests $\Phi$ of level $\leqslant \alpha$.

The works of many authors (see ([1])) lead one to suppose that such a test is the well-known Hotelling test, based on the statistic

$$T^2 = N(N-1)\bar{X}^T S^{-1}\bar{X}. \quad \text{Here } \bar{X} = \frac{1}{N}\sum_{i=1}^{N} X_i, \quad S = \sum_{i=1}^{N}(X_i - \bar{X})(X_i - \bar{X})^T.$$

The test has the form $T^2 \geqslant T_0^2$. In ([1]) this supposition was proved for testing $H_0$ against $H_1$ for $p = 2$ and $N = 3$; in ([2]), for testing $H_0$ against $H_1$ and $H_1'$ for $p = 2$ and $N = 4$. Further progress encounters great analytical difficulties.

In the present note the requirement of minimaxity is replaced by the weaker requirement of "$\varepsilon$-minimaxity" of a test: for any $\varepsilon > 0$, its minimum power on the alternatives $H_1$ or $H_1'$, at a given level $\alpha$ and sample size $N \geqslant N_0(\varepsilon)$, must differ from the maximum of the minimum powers over all tests of level $\leqslant \alpha$ by no more than $\varepsilon$.

Of course, for a given level $\alpha \in (0,1)$ and increasing $N$, only weak signals are of interest, for which $\delta = N\xi^T\Sigma^{-1}$ can grow only slowly; otherwise the power will tend to 1, and the assertion of $\varepsilon$-minimaxity will become trivial.

**Theorem 1.** *The Hotelling test $\Phi_N : T^2 \geqslant T_0^2$ for testing the hypothesis $H_0$ against the hypothesis $H_1$ is $\varepsilon$-minimax for every level $\alpha \in (0,1)$: for any $\varepsilon > 0$ the relation holds*

$$\sup_{\Phi} \inf_{\theta\in H_1} E_\theta \Phi - \inf_{\theta \in H_1} E_\theta \Phi_N \leqslant \varepsilon \tag{1}$$

*for $N > N_0(\varepsilon)$.*

Here $\theta = (\xi, \Sigma)$ is the parameter ($\xi$ is the parameter under study, $\Sigma$ the nuisance parameter); $\Phi$ ranges over all tests of level $\leqslant \alpha$; $E_\theta$ denotes the sign of mathematical expectation.

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* In this form Theorem 1 can apparently also be derived from A. Wald's theorems (see ([3])).

More precise information is given by Theorem 1′.

**Theorem** 1′**.** If the level $\alpha = \alpha_N$ is subject to the conditions

$$O\left(\exp -(\ln N)^{1/6}\right) \leq \alpha \leq 1 - O\left(\exp -(\ln N)^{1/6}\right) \tag{2}$$

and we have

$$\exp\left[-(\ln N)^{1/6}\right] \leq \delta \leq (\ln N)^{1/6}, \tag{3}$$

then

$$\sup_{\Phi} \inf_{\theta\in H_1} E_\theta \Phi - \inf_{\theta \in H_1} E_\theta \Phi_N = O_\varepsilon\left(\frac{1}{N^{1-\varepsilon}}\right) \tag{4}$$

for any $\varepsilon > 0$.

Here $\Phi$ ranges over all tests of level $\leq \alpha$.

**Theorem 2.** The Hotelling test $\Phi_N : T^2 \geq T_0^2$ for testing $H_0$ against the composite hypothesis $H_1'$ is $\varepsilon$-minimax for any fixed level $\alpha \in (0,1)$ and $\delta > 0$: under the conditions of Theorem 1, relation (1) holds with $H_1$ replaced by $H_1'$.

The proof of these theorems is based on the method of N. Giri, J. Kiefer, and C. Stein, developed in their paper [1]. The central place in it is occupied by a lemma on the Giri–Kiefer–Stein integral equation (see [1], p. 1530). This equation has the form

$$\int_{\Gamma_1} \exp\left\{\gamma_1 \sum_{j=1}^{p} \tau_j \sum_{i>j} \beta_i\right\} \prod_{i=1}^{p} \Phi\left(\frac{N-i+1}{2}, \frac{1}{2}\gamma_1\beta_i\tau_i\right) \times \lambda(\beta_1, \dots, \beta_p)\, d\beta = \Phi\left(\frac{N}{2}, \frac{p}{2}, \gamma_1\right), \tag{5}$$

i.e. it is a Fredholm equation of the first kind.

Here $\Gamma_1$ is a simplex; $\beta_i \geq 0$, $\sum_{i=1}^{p} \beta_i = 1$; $\gamma_1 > 0$ is a parameter; $\tau_i \geq 0$, $\sum_{i=1}^{p} \tau_i = 1$, $\tau_i$ are variables; $\lambda(\beta_i, \dots, \beta_p)$ is the desired function, $d\beta$ is an element of Lebesgue measure on the simplex $\Gamma_1$;

$$\Phi(a, c, x) = \sum_{j=0}^{\infty} \frac{\Gamma(a+j)\Gamma(c)}{\Gamma(a)\Gamma(c+j)} \frac{x^j}{j!}$$

is the confluent hypergeometric function (E. Kummer's series). The main lemma states:

**Lemma.** As $N \to \infty$, $\gamma_1 = 2\psi/N$ and $0 \leq \psi \leq 2(\ln N)^{1/3}$, the Giri–Kiefer–Stein equation (5) has the approximate solution

$$\lambda_\infty(\beta_1, \dots, \beta_p) = \frac{\Gamma(p/2)}{(\Gamma(1/2))^p} (\beta_1\beta_2 \cdots \beta_p)^{-1/2}, \tag{6}$$

which is a probability density on the simplex $\Gamma_1$. Namely, when (6) is substituted into the left-hand side of (5), one obtains a residual with the right-hand side of order $O_\varepsilon(N^{-1+\varepsilon})$ for any $\varepsilon > 0$.

Further refinement of this method significantly improves this order and, at the same time, Theorem 1′ (formula (4)).

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## References

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[2] Yu. V. Linnik, V. A. Plys, O. V. Shalaevskii, *DAN*, 168, No. 4 (1966).
[3] A. Wald, *Trans. Am. Math. Soc.*, 54, No. 3, 426 (1943).

*Note: Figure translations are in progress. See original paper for figures.*