EMPTY SPACE-TIME

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Abstract

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PHYSICS

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EMPTY SPACE-TIME

WITH AN ABSOLUTELY PARALLEL VECTOR FIELD

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1. Let V_4 be a space-time of signature $(+---)$ with timelike coordinate x^0. Consider an empty V_4:

$$R_{\alpha\beta} = 0; \qquad \alpha, \beta = 0, 1, 2, 3, \tag{1}$$

possessing an absolutely parallel (i.e., covariantly constant) vector field l^α:

$$l^\alpha{}_{,\beta} = 0. \tag{2}$$

As is known (([3], Ch. II), in order that an empty V_4 belong to type N (the degenerate second type according to Petrov' s classification ([1]), characterized by the vanishing of stationary curvatures), it is necessary and sufficient that there exist a vector field l^α (necessarily isotropic: $l_\alpha l^\alpha = 0$) satisfying equations of the form

$$l^\mu R_{\mu\alpha\beta\nu} = 0 \quad (l^\alpha \neq 0, \quad R_{\mu\alpha\beta\nu} \neq 0), \tag{3}$$

and, as Debever ([4]) showed, for type N such a vector field is unique. Consequently, an empty V_4 with an absolutely parallel vector l^α belongs to type N, and the vector l^α is necessarily isotropic. But in order that V_4 admit a unique isotropic absolutely parallel vector field l^α, it is necessary and sufficient ([5]) that in some coordinate system its metric have the form:

$$g_{\mu\nu} = \begin{pmatrix} \varepsilon & 1 & \varphi & \psi \\ 1 & 0 & 0 & 0 \\ \varphi & 0 & \alpha & \gamma \\ \psi & 0 & \gamma & \beta \end{pmatrix}, \tag{4}$$

where $g_{00} = \varepsilon,\ g_{02} = \varphi,\ g_{03} = \psi,\ g_{22} = \alpha,\ g_{33} = \beta,\ g_{23} = \gamma$ are functions of the variables x^0, x^2, x^3, and

$$l^\alpha = \delta_1^\alpha. \tag{5}$$

Thus, a metric of the form (4), where the functions $g_{\alpha\beta}(x^\sigma)$ satisfy the field equations (1), determines a space-time of type N. This may also be verified directly by determining, for the metric (4), the characteristic of the λ-matrix $(R_{ab} - \lambda g_{ab})$ of the Riemann tensor in the 6-dimensional bivector space under condition (1). Indeed, calculating the components of the tensor $R_{\mu\alpha\beta\nu}$ $(= -R_{\mu\beta\alpha\nu} = -R_{\nu\alpha\beta\mu} = R_{\alpha\mu\nu\beta})$ for the metric (4), it is easy to see that, without taking the field equations into account, only 6 independent components are nonzero: $R_{3232}, R_{3202}, R_{3302}, R_{0202}, R_{0303}, R_{0302}$.

The field equations impose 4 conditions on the metric: they make three components of the tensor $R_{\mu\alpha\beta\nu}$ vanish:

$$R_{3232} = \gamma_{23} - \frac{1}{2}(\alpha_{33} + \beta_{22}) + \frac{1}{4m}\{\alpha_3(\alpha_3\beta - \beta_2\gamma) + \beta_2(\beta_2\alpha - \alpha_3\gamma) +$$

$$+\beta_3[\alpha_2\gamma - \alpha(2\gamma_2 - \alpha_3)] - (2\gamma_3 - \beta_2)[\alpha_2\beta - \gamma(2\gamma_2 - \alpha_3)]\} = 0, \tag{6}$$

$$\begin{aligned} R_{3202} = {} & \frac{1}{2}(\gamma_{02} - \alpha_{03} + \varphi_{23} - \psi_{22}) + \frac{1}{4m}\{(\gamma_0 + \psi_2 - \varphi_3)(\alpha\beta_2 - \gamma\alpha_3) + \\ & + \alpha_0(\beta\alpha_3 - \gamma\beta_2) + (\gamma_0 + \varphi_3 - \psi_2)[\gamma(2\gamma_2 - \alpha_3) - \beta\alpha_2] - \\ & - \beta_0[\alpha(2\gamma_2 - \alpha_3) - \gamma\alpha_2]\} = 0, \\ R_{3302} = {} & \frac{1}{2}(\beta_{02} - \gamma_{03} + \varphi_{33} - \psi_{23}) + \frac{1}{4m}\{(\gamma_0 + \varphi_3 - \psi_2)(\gamma\beta_2 - \beta\alpha_3) + \\ & + \beta_0(\gamma\alpha_3 - \alpha\beta_2) + (\gamma_0 + \psi_2 - \varphi_3)[\alpha\beta_3 - \gamma(2\gamma_3 - \beta_2)] + \\ & + \alpha_0[\beta(2\gamma_3 - \beta_2) - \gamma\beta_3]\} = 0 \end{aligned} \tag{6}$$

and relate the remaining components:

$$
\begin{aligned}
R_{0202} = \varphi_{02} - \frac{1}{2}(\alpha_{00} + \varepsilon_{22}) + \frac{1}{4m}\{(\gamma_0 + \psi_2 - \varphi_3)[\alpha(\gamma_0 + \psi_2 - \varphi_3) - 2\gamma\alpha_0] + \\
+ \beta\alpha_0^2 + (2\psi_0 - \varepsilon_3)[\alpha_2\gamma - \alpha(2\gamma_2 - \alpha_3)] + (2\varphi_0 - \varepsilon_2)[\gamma(2\gamma_2 - \alpha_3) - \beta\alpha_2]\}, \\
R_{0303} = \psi_{03} - \frac{1}{2}(\beta_{00} + \varepsilon_{33}) + \frac{1}{4m}\{(\gamma_0 + \varphi_3 - \psi_2)[\beta(\gamma_0 + \varphi_3 - \psi_2) - 2\gamma\beta_0] + \\
+ \alpha\beta_0^2 + (2\psi_0 - \varepsilon_3)[\gamma(2\gamma_3 - \beta_2) - \alpha\beta_3] + (2\varphi_0 - \varepsilon_2)[\gamma\beta_3 + \beta(\beta_2 - 2\gamma_3)]\}, \\
R_{0302} = \frac{1}{2}(\psi_{02} + \varphi_{03} - \varepsilon_{23} - \gamma_{00}) + \frac{1}{4m}\{(\gamma_0 + \varphi_3 - \psi_2)[\beta\alpha_0 - \\
- \gamma(\gamma_0 + \psi_2 - \varphi_3)] + \beta_0[\alpha(\gamma_0 + \psi_2 - \varphi_3) - \gamma\alpha_0] + \\
+ (2\varphi_0 - \varepsilon_2)(\gamma\beta_2 - \beta\alpha_3) + (2\psi_0 - \varepsilon_3)(\gamma\alpha_3 - \alpha\beta_2)\}
\end{aligned}
$$

by the relation

$$\alpha R_{0303} + \beta R_{0202} - 2\gamma R_{0302} = 0, \tag{7}$$

where the notation has been introduced: $f_\alpha \equiv \partial f / \partial x^\alpha$ $(\alpha = 0, 1, 2, 3)$,

$$m \equiv -\det \|g_{\alpha\beta}\| = \alpha\beta - \gamma^2 > 0. \tag{8}$$

Using the field equations (6)–(7) and condition (8), one can show that the desired characteristic has the form $[(21, 21)^0]$, i.e., determines type N, which proves the assertion.

Thus, the metric (4), where the six functions $g_{\alpha\beta}(x^\sigma)$ satisfy the four equations (6) and (7), determines a solution of the Einstein equations (1) belonging to type N of Petrov' s classification. It is easy to show that many known solutions of equations (1) ([2], [6], [7]); ([1]), pp. 230–234, etc.), physically interpreted as describing gravitational waves, are particular cases of the given solution.

2. From the form of the metric (4) it follows directly that V_4 with the given metric admits a one-parameter group of motions G_1 with operator

$$X = l^\alpha \partial / \partial x^\alpha = \partial / \partial x^1. \tag{9}$$

We assert that, for a metric of the general form (4) in vacuum, the group G_1 is the complete group of motions. To prove this it is sufficient to show that the rank ρ of the system of equations

$$\mathcal{L} R_{\mu\alpha\beta\nu} = 0; \quad \mathcal{L} R_{\mu\alpha\beta\nu,\lambda_1} = 0; \dots; \quad \mathcal{L} R_{\mu\alpha\beta\nu,\lambda_1 \dots \lambda_p} = 0; \dots \tag{10}$$

($\mathcal{L}$ denotes the Lie derivative in the direction of the vector ξ^α) with respect to the p-th order in the unknowns ξ_α and $\xi_{\alpha,\beta}$, under the conditions $\xi_{(\alpha,\beta)} = 0$, for the metric (4) in vacuum is equal to 9, and the addition of the equations from (10) with number $p+1$ does not change this rank ([1], p. 82). Since one solution of the Killing equations for the metric (4) exists and is determined by the operator (9), it follows that $\rho \leq 9$. It is easy to see that for the metric (4), under condition (1), the only nonzero components are $R_{0202,\tau}$; $R_{0303,\tau}$; $R_{0302,\tau}$ $(\tau = 0, 2, 3)$ of the tensor $R_{\mu\alpha\beta\nu,\sigma}$, and the components $R_{0202,\tau\varepsilon}$; $R_{0303,\tau\varepsilon}$; $R_{0302,\tau\varepsilon}$ $(\tau, \varepsilon = 0, 2, 3)$ of the tensor $R_{\mu\alpha\beta\nu,\sigma\rho}$. Substituting them into equations (10), we are convinced that the first two systems (10) contain 9 independent equations, which proves our assertion.

In view of condition (2), the vector l^α in space-time with metric (4) in vacuum is uniquely associated with a family of isotropic three-dimensional hypersurfaces ${}^*V_3$:

$$f(x^\alpha) = \text{const}, \tag{11}$$

where the scalar f is defined by the equality $l_\alpha = \partial_\alpha f$. Then, by virtue of (5), in the coordinate system under consideration (isotropic-semigeodesic, see ([1]), p. 56) this family is represented by the equation $x^0 = \text{const}$, and the trajectories of the operator (9) by coordinate lines x^1 orthogonal to it. It follows from (4) and (8) that the rank of the metric of any hypersurface of the family (11) is 2. This means that the family (11) determines a unique congruence of **special lines** ([8]), isotropic and orthogonal to any hypersurface of the family. By virtue of the uniqueness of the absolutely parallel vector field l^α, the trajectories of the operator (9) coincide with this congruence, and, consequently, the operator (9) is special ([8]). Moreover, since the trajectories of an absolutely parallel vector form a congruence of geodesic lines, the trajectories of the operator of the group G_1, admitted by the general metric of the form (4) in vacuum, form a congruence of isotropic geodesics of the metric (4).

As is known (([1]), pp. 175–179), space-time in vacuum can admit only a group of motions which is a subgroup of the group of maximal mobility of the corresponding type of gravitational fields. In this case the operator of the group G_1 enters into the basis of the operators of the group G_r $(r > 1)$. Thus, an empty space-time with a metric belonging to the class of metrics (4) admits a group of motions G_r $(1 \leq r \leq 6)$ containing the special operator (9). The trajectories of this operator form a congruence of isotropic geodesics of this metric.

3. In ([9]) it was shown that every empty V_4 of type N satisfies equations of the form

$$R_{\mu\alpha\beta\nu,\sigma} = 0. \tag{12}$$

Conversely, every nonsymmetric Einstein space $(R_{\alpha\beta} = \varkappa g_{\alpha\beta};\ R_{\mu\alpha\beta\nu,\sigma} \neq 0)$ satisfying equations (12) is an empty V_4 of type N. This makes it possible to regard empty V_4 of type N as fields of pure gravitational radiation. A deeper physical interpretation is admitted by the class of empty gravitational fields with metric of type (4). It is easy to see that the family of isotropic hypersurfaces $^*V_3$ (11) is a family of characteristic hypersurfaces of the Einstein equations (1) ($g^{\alpha\beta} f_{,\alpha} f_{,\beta} = 0$), as a result of which each $^*V_3$ of the family (11) is a **front of a gravitational wave**, determined by specifying the vector field l^α at each point. The trajectories of the special operator (9) of the group of motions admitted by the metric of class (4) coincide with the bicharacteristics of the Einstein equations ($\dot{x}^\alpha = g^{\alpha\sigma} f_{,\sigma}$) and, consequently, are the directions of propagation of the gravitational wave–**gravitational rays**. As is easy to see, the congruence of rays is identical with the totality of all one-dimensional surfaces of transitivity of the group G_1 with the special operator (9).

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Note: Figure translations are in progress. See original paper for figures.