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Abstract

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V. I. YUDOVICH

ON THE BIFURCATION OF ROTATIONAL FLOWS OF A LIQUID

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In paper ([1]) the question of bifurcation of stationary flows of a viscous incompressible liquid was considered, and an example was given of nonuniqueness of the solution of the stationary problem for the Navier–Stokes equations. In the present note the birth of secondary stationary flows is studied in a liquid enclosed between cylinders rotating (in the same direction).

1. Let a homogeneous viscous incompressible liquid be enclosed in the cavity between coaxial cylinders $r = r_1$ and $r = r_2$ (r, θ, z are cylindrical coordinates). We pose the problem of finding axisymmetric stationary flows, i.e., such that the velocity components v_r', v_θ', v_z' depend only on r and z. We shall also assume that v_r', v_θ', v_z' are periodic in z with period $2\pi/a_0$ and that the flux of velocity through a transverse section of the cavity is equal to zero:

$$\int_{r_1}^{r_2} v_r'(r, z)\, r\, dr = 0. \tag{1}$$

Let the cylinders be solid and rotate with angular velocities ω_1, ω_2, respectively, and let the vector of vortical body forces have the form $\left(0, \frac{1}{\lambda}F(r), 0\right)$ (where λ is the Reynolds number). Then it is easy to verify that one of the solutions of the posed problem is the flow with velocity vector $\mathbf{v}_0 = (0, v_0(r), 0)$, where the function $v_0(r)$ is uniquely determined as the solution of the boundary-value problem

$$\left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{1}{r^2}\right) v_0 = -F(r); \qquad v_0(r_1) = \omega_1 r_1; \quad v_0(r_2) = \omega_2 r_2. \tag{2}$$

In particular, if $F = 0$, the well-known Couette flow is obtained,

$$v_0(r) = ar + \frac{b}{r}; \qquad a = \frac{\omega_2 r_2^2 - \omega_1 r_1^2}{r_2^2 - r_1^2}; \qquad b = \frac{(\omega_1 - \omega_2) r_1^2 r_2^2}{r_2^2 - r_1^2}. \tag{3}$$

Seeking solutions $\mathbf{v}'$ of the posed problem, different from $\mathbf{v}_0$, in the form $\mathbf{v}' = \mathbf{v} + \mathbf{v}_0$, we obtain a system of equations for the vector $\mathbf{v}$

$$\frac{1}{r}\frac{\partial}{\partial r}(rv_r) + \frac{\partial v_z}{\partial z} = 0; \tag{4}$$

$$\Delta v_r - \frac{v_r}{r^2} - \frac{\partial p}{\partial r} = \lambda\left[v_r\frac{\partial v_r}{\partial r} + v_z\frac{\partial v_r}{\partial z} + \frac{v_\theta^2}{r} - 2\omega v_\theta\right]; \tag{5}$$

$$\Delta v_\theta - \frac{v_\theta}{r^2} = \lambda\left[v_r\frac{\partial v_\theta}{\partial r} + v_z\frac{\partial v_\theta}{\partial z} + \frac{v_r v_\theta}{r} - g v_r\right]; \tag{6}$$

$$\Delta v_z - \frac{\partial p}{\partial z} = \lambda\left[v_r\frac{\partial v_z}{\partial r} + v_z\frac{\partial v_z}{\partial z}\right], \tag{7}$$

$$\omega = \frac{v_0}{r}; \qquad g = -\left(\frac{dv_0}{dr} + \frac{v_0}{r}\right).$$

Here the functions v_r, v_θ, v_z must be $2\pi/a_0$-periodic in z, vanish for $r = r_1, r_2$, and satisfy the condition of the form (1).

Let M be the set of twice continuously differentiable solenoidal vectors defined in the closed domain $\{r_1 \le r \le r_2;\ -\infty < z < \infty\}$, axisymmetric, vanishing for $r = r_1, r_2$, having zero flux through a cross-section of the cavity, and such that v_r, v_θ are even functions of z, while v_z is odd. By H_1^0 we denote the Hilbert space obtained by completing M in the norm generated by the scalar product

$$(\mathbf{u}, \mathbf{v})_{H_1^0} = -\int_{-\pi/\alpha_0}^{\pi/\alpha_0} dz \int_{r_2}^{r_2} \Delta\mathbf{v}\cdot\mathbf{u}\, r\, dr. \tag{8}$$

Inverting the operator defined by relations (4)–(7) for $\lambda = 0$, we reduce problem (4)–(7) to the operator equation

$$\mathbf{v} = \lambda K_0 \mathbf{v}. \tag{9}$$

As was shown in [1], the operator K_0 is completely continuous in H_1^0 and has Fréchet differential A_0 at the point $\mathbf{v} = 0$. The operator equation

$$\mathbf{u} = \lambda A_0 \mathbf{u} \tag{10}$$

is equivalent to a system of the form (4)–(7) with the nonlinear terms omitted. The adjoint equation

$$\mathbf{w} = \lambda A_0^* \mathbf{w} \tag{11}$$

is equivalent to an analogous system in which only the functions 2ω and g are interchanged.

Theorem 1. *Suppose that the conditions*

$$\omega(r) = v_0/r > 0 \qquad (r_1 < r < r_2); \tag{12}$$

$$g(r) = -(dv_0/dr + v_0/r) > 0 \qquad (r_1 < r < r_2). \tag{13}$$

are satisfied. Then for any α_0, with the exception of at most some countable set, the spectrum of the operator A_0 consists of a sequence of simple positive characteristic values $0 < \lambda_1 < \lambda_2 < \cdots$ and of their opposites $-\lambda_1, -\lambda_2, \ldots$. Each of them is a bifurcation point of equation (9). All intervals $(\lambda_1, \lambda_2), (\lambda_3, \lambda_4), \ldots$ and the intervals symmetric to them belong to the spectrum of equation (9), i.e. for any λ from these intervals equation (9) (and together with it problem (4) –(7)) has nontrivial solutions.

We shall set forth the main points of the proof of this theorem. Every solution of equation (10) is a linear combination of solutions of the form

$$u_r = u(r)\cos\alpha z; \qquad u_\theta = v(r)\cos\alpha z; \qquad u_z = w(r)\sin\alpha z, \tag{14}$$

where $\alpha = k\alpha_0$ (k is a natural number), $w = -\dfrac{1}{\alpha r}\dfrac{d}{dr}ru$, and the functions u, v are determined by solving the boundary-value problem

$$(L - \alpha^2)^2 u = 2\alpha^2 \lambda \omega v; \tag{15}$$

$$(L - \alpha^2) v = -\lambda g u, \tag{16}$$

where

$$L = \frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{1}{r^2}.$$

The boundary conditions at the ends $r = r_1, r_2$ have the form

$$u = u' = 0; \tag{17}$$

$$v = 0. \tag{18}$$

Introduce the Green functions $G_1(r,\rho)$, $G_2(r,\rho)$ of the differential operators $-r(L-\alpha^2)$, $r(L-\alpha^2)^2$ under the boundary conditions (18) and (17), respectively. We define the Green operators G_1, G_2 by the formula

$$G_k f = \int_{r_1}^{r_2} G_k(r,\rho) f(\rho)\rho\, d\rho \qquad (k = 1, 2). \tag{19}$$

Let H_0 be the Hilbert space L_2 with weight r on the interval (r_1, r_2). The operators G_k are completely continuous, symmetric, and positive in H_0.

Lemma 1. *The kernels $G_1(r,\rho)$, $G_2(r,\rho)$ are symmetric oscillation kernels.*

This lemma is easily proved by the methods of M. G. Krein ([2]) (see also ([3])).

The spectral problem (15)–(18) is equivalent to either of the integral equations ($\mu = 2\alpha^2\lambda^2$)

$$u = \mu G_2 \omega G_1 g u; \tag{20}$$

$$v = \mu G_1 g G_2 \omega v. \tag{21}$$

It follows from Lemma 1 that the kernels of the integral operators in (20) and (21) are oscillatory. Applying the results from ([4]), we arrive at the following assertion.

Lemma 2. *The spectrum of equations (20) and (21) consists of a sequence of simple positive characteristic numbers*

$$0 < \mu_1(\alpha) < \mu_2(\alpha) < \cdots.$$

By Lemma 2, the spectrum of the operator A_0 consists of real characteristic numbers

$$\lambda_{ik} = \sqrt{\mu_i(k\alpha_0)}/2k^2\alpha_0^2, \qquad \lambda'_{ik} = -\lambda_{ik} \quad (i, k = 1, 2, ...).$$

In what follows we shall speak only about the characteristic numbers λ_{ik}; the numbers λ'_{ik} are considered analogously.

Lemma 3. *The rank of each of the characteristic numbers λ_{ik}, λ'_{ik} of the operator A_0 is equal to 1.*

This lemma is proved with the aid of Lemma 2 and Lemma 1.5 from ([1]).

Lemma 4. *The operators G_1 and G_2 are analytic in α in some strip $|\operatorname{Im}\alpha| < \delta_0$.*

From Lemmas 2 and 4, by the method of perturbation theory, we derive that the functions $\mu_1(\alpha), \mu_2(\alpha), ...$ are analytic on the real axis, and the functions

$$\Lambda_i(\alpha) = \sqrt{\mu_i(\alpha)}/\alpha^2$$

are analytic on the half-axis $\alpha > 0$. In view of Lemma 3, the multiplicity of the characteristic number λ_{ik} is equal to the dimension of the corresponding subspace of eigenfunctions. The latter, obviously, is equal to the number of numbers λ_{ir} coinciding with λ_{ik}. We shall show that for any α_0, except for some countable set Γ, all the eigenvalues λ_{ik} $(i, k = 1, 2, ...)$ are distinct. Indeed, the set Γ is the union of the sets Γ_{ikrs}—those α_0 for which the equality

$$\Lambda_i(k\alpha_0) - \Lambda_r(s\alpha_0) = 0 \qquad (i, k, r, s = 1, 2, ...) \tag{22}$$

is satisfied. But Γ_{ikrs}, as the set of zeros of the analytic function

$$\varphi_{ikrs}(\alpha) = \Lambda_i(k\alpha) - \Lambda_r(s\alpha)$$

(it is easy to prove that $\varphi_{ikrs} \not\equiv 0$), is at most countable, and therefore the set Γ is at most countable. (In fact, Γ_{ikrs} is countable, its limit points being 0 and ∞.)

Renumber the numbers λ_{ik} in increasing order; we obtain the sequence

$$0 < \lambda_1 < \lambda_2 < \cdots.$$

It remains only to apply the theorem of M. A. Krasnosel' skii on bifurcation points ([5]) and to note that for

$$\lambda \in (\lambda_1, \lambda_2) \cup (\lambda_3, \lambda_4) \cup \cdots$$

the index of the fixed point $v = 0$ of equation (9) is equal to -1, whereas the rotation of the vector field $(I - \lambda K_0)v$ on large spheres of the space H_1^0 is equal to $+1$ (see ([6])). This completes the proof of the theorem.

In the case of Couette flow (3), conditions (12), (13) lead to the requirement that the cylinders rotate in the same direction:

$$\omega_1 > 0, \qquad \omega_2 \geq 0$$

(the outer cylinder may be stationary), and that the inequality

$$\omega_2 r_2^2 - \omega_1 r_1^2 < 0 \tag{23}$$

be satisfied.

If (23) is not satisfied, then Couette flow is stable for all Reynolds numbers ([7]).

2. The question of the stability of the flow $\mathbf{v}_0$ reduces, as is known, to the study of the spectrum of the boundary-value problem

$$(L-\alpha^2)^2 u - \sigma(L-\alpha^2)u = 2\alpha^2\lambda\omega v, \tag{24}$$

$$(L-\alpha^2)v - \sigma v = -\lambda g u \tag{25}$$

with the boundary conditions (17), (18). It turns out that conditions (12), (13) imply instability of the flow $\mathbf{v}_0$.

Theorem 2. *Suppose that conditions (12), (13) are satisfied. Then, for every* $\sigma \geq -(a^2+\sigma_0)$*, there corresponds a sequence of values* λ*:* $0 < \lambda_1 < \lambda_2 < \ldots$ *such that problem (24)–(25) has a nontrivial solution. Here* $\sigma_0 > 0$ *and depends only on* r_1, r_2*.*

Let us note that the instability of Couette flow (i.e., the presence in the spectrum of problem (24)–(25) of eigenvalues σ with $\operatorname{Re}\sigma > 0$) was proved in [8]. The validity of linearization in the nonlinear problem of hydrodynamic stability was substantiated in [9].

Acting by the method of perturbations, one can consider various "nearby" problems, for example noncoaxial cylinders, the case of small $\omega_2 < 0$, etc. One can also, by imposing small time-periodic perturbations on the mass forces and on the angular velocities of rotation of the cylinders, obtain examples of nonuniqueness of the solution of the problem of forced periodic oscillations of a fluid [10].

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Note: Figure translations are in progress. See original paper for figures.