\(L_p\)-ESTIMATES OF A CERTAIN CLASS OF ANISOTROPICALLY SINGULAR INTEGRALS

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Abstract

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O. V. BESOV, V. P. IL' IN, P. I. LIZORKIN

L_p-ESTIMATES OF A CERTAIN CLASS OF ANISOTROPICALLY SINGULAR INTEGRALS

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1. In the paper (1) of Zygmund and Calderón, estimates are given for singular integrals of the form

$$\lim_{\varepsilon\to 0}\int_{|x-y|>\varepsilon} k(x-y)f(y)\,dy,$$

where $k(x)$ is a homogeneous function of degree $-n$, satisfying Dini' s condition on the sphere $S\{x;\ |x|=1\}$ and with zero mean value on this sphere. The kernel $k(x)$ is written in the form

$$k(x)=k(x/|x|)/|x|^n=k(u)/r^n,\quad u\in S;$$

it is isotropic in the sense that the character of its decrease in any direction is one and the same. This property is not possessed by singular kernels arising in the differentiation of elementary solutions of quasi-elliptic equations. As a simple example of an anisotropic singular kernel for $n=2$, the function

$$k(x,y)=y/(x^2+\sqrt{x^4+4y^2})^{5/2} \tag{1}$$

may serve.

In the present paper we use a certain generalization of the concept of homogeneity of a function (see Definition 1). For a function $k(x)$ homogeneous in the generalized sense of degree $-n$, a regularization process is constructed for the convolution $k*f$, and it is proved that the operator $K: f\to k*f$ arising in this way is bounded in L_p. The resulting Theorem 1 generalizes the corresponding result of Zygmund-Calderón (1) to the case of anisotropic kernels of a definite form. The concluding theorems also make it possible to treat nonhomogeneous kernels and other regularization methods. Similar questions were considered in ($^{2-4}$).

2. **Definition 1.** Let a vector $a=(a_1,\dots,a_n)$, $a_j>0$, $j=1,\dots,n$, $\sum_{j=1}^n a_j=n$, be given. A single-valued function $k(x)=k(x_1,\dots,x_n)$, defined in the

n-dimensional space E_n, is called an a-homogeneous function of degree m if

$$k(t^a x) \equiv k(t^{a_1} x_1, \dots, t^{a_n} x_n) = t^m k(x)$$

for every $t > 0$. For example, the function (1) is a-homogeneous of degree -2, $a = (2/3, 4/3)$.

Theorem 1. Let the a-homogeneous function $k(x)$ of degree $-n$ satisfy the conditions

$$\int_S k(x) \sum_{j=1}^{n} a_j x_j^2 \, dS = 0, \tag{2}$$

$$|k(x) - k(y)| \leqslant \omega(|x-y|), \quad x, y \in S, \quad \int_0^1 \frac{\omega(t)}{t} \, dt < \infty. \tag{3}$$

Then the operator K_ε, defined by the formula

$$K_\varepsilon f = \int_{\sum_1^n \frac{(x_j - y_j)^2}{\varepsilon^{2a_j}} > 1} k(x-y) f(y) \, dy,$$

is bounded in L_p, $1 < p < \infty$, i.e.

$$\|K_\varepsilon f\|_{L_p} \leq c \|f\|_{L_p}, \tag{4}$$

where the constant c does not depend on ε. Moreover, in the sense of convergence in L_p we have

$$\lim_{\varepsilon \to 0} K_\varepsilon f = K f, \tag{5}$$

and the operator K thus defined is bounded in L_p.

We note that for $a_1 = \cdots = a_n = 1$ the theorem just formulated becomes the Zygmund–Calderón theorem from (1). The theorem is applicable to the kernel $k(x, y)$ of example (1).

3. We outline the method and the scheme of the proof of Theorem 1.

Definition 2. An a-homogeneous function $\eta(x)$ of the first degree, positive and continuous for $x \neq 0$, is called the η-distance of the point x from the origin.

Definition 3. The line $\mathcal{L}^u$ defined by the equations $x_j = u_{jt}^{a_j}$, $j = 1, \dots, n$, where the numbers $u_1, \dots, u_n$ are fixed and the parameter t varies from 0 to ∞, will be called the a-trajectory passing through the point u.

Between the points of the unit level surface S_1^η of the η-distance, determined by the equation $\eta(x) = 1$, and all possible a-trajectories there is a one-to-one correspondence established by the formulas $x_j = u_j \eta^{a_j}$, $j = 1, \dots, n$, $u \in S_1^\eta$. These formulas give the passage from Cartesian coordinates to "η-spherical coordinates" (η, u). In our considerations the following η-distances were used: 1) $\eta_1(x) = \pi(x) = \max_j\{|x_j|^{\alpha_j}\}$, $\alpha_j = 1/a_j$; S_1^π coincides with the surface of the unit cube, the inequality $\pi(x) < t$ defines the parallelepiped $\Pi_t\{x;\ |x_j| < t^{a_j},\ j = 1, \dots, n\}$; 2) $\eta_2(x) = \rho(x)$, where the positive function $\rho(x)$ is defined implicitly by the equation

$$\sum_1^n x_j^2/\rho^{2a_j} = 1;$$

$S_1^\rho = S$ coincides with the unit sphere, and the inequality $\rho(x) < t$ defines the ellipsoid

$$\mathcal{E}_t\left\{x; \sum_1^n x_j^2/t^{2a_j} < 1\right\}.$$

We note that in ρ-spherical coordinates an a-homogeneous function is written in the form

$$k(x) = k(x_1/\rho^{a_1}, \dots, x_n/\rho^{a_n})/\rho^n(x) = k(u)/\rho^n, \quad u \in S.$$

The function $k(x_1/\rho^{a_1}(x), \dots, x_n/\rho^{a_n}(x))$ is a-homogeneous of degree zero; it is determined by its values on the unit sphere and is constant along each trajectory. We shall call $k(u)$ the characteristic.

The passage from integration in Cartesian coordinates to integration in ρ-spherical coordinates is effected by the formula

$$\int f(x)\,dx = \int f(u_1\rho^{a_1}, \dots, u_n\rho^{a_n}) \sum_{j=1}^n a_j u_j^2 \rho^{n-1}\,d\rho\,dS.$$

In proving Theorem 1 we follow the scheme of Hörmander (5), replacing homogeneity by a-homogeneity and spherical coordinates by ρ-spherical coordinates. We also make essential use of the interpolation theorem given by Kreĭn in the paper (3).*

A. The truncated kernel $k_\varepsilon(x)$ is considered; it coincides with $k(x)$ for $\rho(x) > \varepsilon > 0$, is equal to zero for $\rho(x) \le \varepsilon$, and it is established that there exist—

* We formulate it below in the form needed.

there exist constants $N > 1$ and c, independent of ε, such that for any $t > 0$

$$\int_{\pi(x) > Nt} |k_\varepsilon(x - y) - k_\varepsilon(x)|\,dx \le c \quad \text{if } \pi(y) < t. \tag{6}$$

Here one has to prove that if $x_j = u_j\rho^{a_j}$, $x_j - y_j = v_j\tau^{a_j}$, $u, v \in S$, then $|u - v| \leq c(t/\rho)^{\min_j a_j}$, and to introduce condition (3).

B. Using inequality (6) and condition (2), in the usual way $^{(5)}$ we obtain an estimate for the Fourier transform $\tilde{k}_\varepsilon(\lambda)$ of the truncated kernel: $|\tilde{k}_\varepsilon(\lambda)| \leq c$, where c is independent of ε. It follows from this that the convolution $k_\varepsilon * f$ is a bounded operator in L_2.

C. The above-mentioned interpolation theorem of Krée states that if, for a locally summable kernel h, property (6) is satisfied and the convolution $h * f$ is bounded in some L_{p_0} ($\|h * f\|_{L_{p_0}} \leq c\|f\|_{L_{p_0}}$, $1 < p_0 < \infty$), then $\|h * f\|_{L_p} \leq MC\|f\|_{L_p}$, $1 < p < \infty$, where M does not depend on h. In our case $h = k_\varepsilon$ and $p_0 = 2$. Assertion (4) is thus proved.

D. Passing to the limit as $\varepsilon \to 0$, we obtain (5). Let us explain the role of condition (2). On the basis of the relation $\tilde{k}_\varepsilon(\lambda) = \tilde{k}_1(\varepsilon^a\lambda)$, we write

$$\tilde{k}_1(\lambda) - \tilde{k}_1(\varepsilon^a\lambda) = \frac{1}{(2\pi)^{n/2}} \int_{\varepsilon<\rho(x)<1} k(x)e^{-i\lambda x}\,dx =$$

$$= c\int_S k(u)\sum_1^n a_j u_j^2\,du \int_\varepsilon^1 \frac{\exp\left[-i\sum_1^n \lambda_j u_j \rho^{a_j}\right]}{\rho}\,d\rho.$$

Consequently,

$$\lim_{\lambda\to 0}[\tilde{k}_1(\lambda) - \tilde{k}_1(\varepsilon^a\lambda)] = \frac{1}{(2\pi)^{n/2}} \ln\frac{1}{\varepsilon} \int_S k(u)\sum a_j u_j^2\,du.$$

Letting ε tend to 0, we find that if $\tilde{k}$ is a bounded function, then (2) is necessarily satisfied. We proceed further in the standard way $^{(5)}$.

4. The following theorem generalizes the theorem of Lewis announced in $^{(4)}$ (p. 548).

Theorem 2. Let the function $k(x)$ be locally summable in $E_n - \{0\}$, and let there exist constants N and C such that:

I.

$$\int_{\pi(x)>Nt} |k(x-y) - k(x)|\,dx \leq c \quad \text{if } \pi(y) \leq t, \quad 0 < t < \infty.$$

II.

$$\int_{t<\pi(x)<Nt} |k(x)|\,dx \leq c, \quad 0 < t < \infty.$$

Suppose, moreover, that there exists an increasing family of domains $\mathcal{M}_t$ such that $\Pi_t \subset \mathcal{M}_t \subset \Pi_{Nt}$, for which, for any t_1, t_2, $0 < t_1 < t_2 < \infty$,

III.

$$\int_{M_{t_2}-\mathcal{M}_{t_1}} k(x)\,dx = 0.$$

If $k_{\varepsilon\nu}(x)$ coincides with $k(x)$ for $x \in \mathcal{M}_\nu - \mathcal{M}_\varepsilon$ and is equal to zero for $x \notin \mathcal{M}_\nu - \mathcal{M}_\varepsilon$, then:

1) $\|\tilde{k}_{\varepsilon\nu}\| \leq MC$, where M does not depend on $k_{\varepsilon\nu}$.
2) The operator $K_{\varepsilon\nu}$ of convolution with the kernel $k_{\varepsilon\nu}$ is bounded in L_p; $\lim_{\varepsilon\to 0,\ \nu\to\infty} K_{\varepsilon\nu} f = Kf$ exists in the sense of convergence in p-mean on every compact set, and the operator K thus defined is bounded in L_p, $1 < p < \infty$.

Proof. Consider the family of functions

$$k^{(s)}(x) = s^n k(s^{a_1}x_1, \dots, s^{a_n}x_n), \quad 0 < s < \infty.$$

Fix s and put $x_j = x'_j s^{a_j}$. For

under this mapping Π_t goes into $\Pi_{t/s}$, Π_{Nt} into $\Pi_{Nt/s}$, $\mathcal{M}_t$ into $\mathcal{M}^{(s)}_{t/s}$. Although the domains $\mathcal{M}^{(s)}_\tau$ do not coincide with $\mathcal{M}_\tau$, nevertheless the inclusions $\Pi_\tau \subset \mathcal{M}^{(s)}_\tau \subset \Pi_{N\tau}$, $0 < \tau < \infty$, are valid. A simple calculation shows that the function $k^{(s)}(x)$ satisfies conditions I, II of Theorem 2 and condition III with $\mathcal{M}_t$ replaced by $\mathcal{M}^{(s)}_\tau$.

For any fixed ε, ν, and $\lambda \neq 0$, we have

$$\tilde{k}_{\varepsilon\nu}(\lambda) = \frac{1}{(2\pi)^{n/2}} \int_{M_\nu - \mathcal{M}_\varepsilon} k(x) e^{-i\lambda x}\,dx = \frac{1}{(2\pi)^{n/2}} \int_{M^{(s)}_{\nu/s} - \mathcal{M}^{(s)}_{\varepsilon/s}} e^{-i(\lambda' x')} k^{(s)}(x')\,dx', \tag{7}$$

where $\lambda' = (\lambda_1 s^{\alpha_1}, \dots, \lambda_n s^{\alpha_n})$. Choose s so that $|\lambda'| = 1$. If $\nu/s = \nu' \leq 1$, then, using condition III, we write

$$\tilde{k}_{\varepsilon\nu}(\lambda) = \frac{1}{(2\pi)^{n/2}} \left| \int_{M^{(s)}_{\nu'} - \mathcal{M}^{(s)}_{\varepsilon'}} (e^{-i(x'\lambda')} - 1) k^{(s)}(x')\,dx' \right| \leq \int_{M^{(s)}_{\nu'}} |x'|\,|k^{(s)}(x')|\,dx.$$

Hence we obtain $|\tilde{k}_{\varepsilon\nu}(\lambda)| \leq MC$. Next let $\varepsilon/s = \varepsilon' \geq 1$. Denote $(k^{(s)})_{\varepsilon'\nu'} = k^{(s)}_{\varepsilon'\nu'}$; by equality (7) we have $k_{\varepsilon\nu}(\lambda) = \tilde{k}^{(s)}_{\varepsilon'\nu'}$. The relation is obvious

$$(e^{-i(y'\lambda')}-1)\tilde{k}^{(s)}_{\varepsilon'\nu'}(\lambda') = \frac{1}{(2\pi)^{n/2}}\int_{E_n} e^{-i(x'\lambda')}[k^{(s)}_{\varepsilon'\nu'}(x'-y') - k^{(s)}_{\varepsilon'\nu'}(x')]\,dx'.$$

Hence, relying on I, II and since $|\lambda'| = 1$, we obtain the estimate $|k^{(s)}_{\varepsilon'\nu'}(\lambda')| \leq MC$. Finally, in the case when $\varepsilon < s < \nu$, we use the equality $\tilde{k}_{\varepsilon\nu} = \tilde{k}_{\varepsilon s} + \tilde{k}_{s\nu}$. Assertion 1) is proved.

Now we draw the conclusion about the boundedness of the operator $K_{\varepsilon\nu}f = k_{\varepsilon\nu} * f$ in L_2, verify the fulfillment (uniformly in ε, ν) of the conditions of the theorem for the function $k_{\varepsilon\nu}$, use the interpolation theorem, and carry out the passage to the limit as $\varepsilon \to 0$, $\nu \to \infty$. We obtain assertion 2).

5. With the help of Theorem 2 one can indicate other (different from truncation) methods of regularizing the convolution $k * f$.

Theorem 3. *Let a function $k(x)$, satisfying the conditions of Theorem 2, be uniformly approximated on each compact subset of $E_n - \{0\}$ as $\sigma \to 0$ by locally summable functions $k(x,\sigma)$. If, for each σ, the function $k(x,\sigma)$ satisfies conditions I and II (where the choice of C and N does not depend on σ) and condition III (with domains $\mathcal{M}^{\sigma}_t$, for which $\mathcal{M}^{\sigma}_t = \mathcal{M}_t$ for large t and $\Pi_t \subset \mathcal{M}^{\sigma}_t \subset \Pi_{Nt}$). Then, in the sense of convergence in the p-mean on every compact set,*

$$\lim_{\sigma\to 0}\int_{E_n} k(x-y,\sigma)f(y)\,dy = Kf$$

and the operator K is bounded in L_p, $1 < p < \infty$.

V. A. Steklov Mathematical Institute
Academy of Sciences of the USSR

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Note: Figure translations are in progress. See original paper for figures.

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