SOME QUESTIONS IN THE THEORY OF VECTOR INVARIANTS

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SOME QUESTIONS IN THE THEORY OF VECTOR INVARIANTS

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1. Let P be an algebra over a field R; let G be some group of its automorphisms. We shall denote by $I(P,G)$ the subalgebra of G-invariant elements.

The following questions are of interest:

A. For which P and G does the algebra $I(P,G)$ have a finite number of generators (is finitely generated)?

B. How many algebraically independent elements are there in $I(P,G)$?

C. How are the algebras of invariants of the group G and of its subgroup H related?

In the case when P is a polynomial algebra and G is a group of linear transformations, these questions were considered in the classical theory of invariants, with question A being known as Hilbert' s 14th problem ([1]).

In the present work the following has been done. Question A is answered positively for a certain class of Lie groups, broader than the class of reductive Lie groups (Theorem 3); this class contains, in particular, maximal nilpotent and maximal solvable subgroups of reductive groups. Further, for almost all semisimple groups it is proved that the dimension of an orbit of maximal dimension is equal to the dimension of the group (Theorem 4); from this one obtains a solution of question B for such groups (a corollary of Theorem 4). Question C is solved for the case when G is an arbitrary connected Lie group, and H is its maximal nilpotent subgroup (Theorem 2). These results are new also in the classical case: P is a polynomial algebra, G is a group of linear transformations.

For simplicity, in what follows the field of complex numbers is taken as the ground field; all results carry over without change to the case of an arbitrary algebraically closed field of characteristic zero.

2. A universal representation of a connected Lie group G is a completely reducible representation of the group G in a space U such that: 1) every irreducible subspace is finite-dimensional and 2) every finite-dimensional

irreducible representation of the group G is contained in U once. If, moreover, U is a commutative associative algebra with identity and without zero divisors, and G is a group of its automorphisms, then we shall call U a universal algebra of the group G and denote it by $U(G)$.

Theorem 1. *Every connected Lie group possesses a universal algebra.*

The proof is based on a construction of D. P. Zhelobenko (^{2}).

Theorem 2. *Let G be a connected Lie group of automorphisms of an algebra P; let H be a certain maximal connected nilpotent subgroup of G; let $P' = U(G) \otimes P$ be the tensor product of the algebras $U(G)$ and P. Suppose that every subspace of P irreducible with respect to G is finite-dimensional. Then the algebras $I(P', G)$ and $I(P, H)$ are isomorphic.*

3. For brevity, we shall call an associative algebra P with identity an S-algebra if it is: 1) finitely generated; 2) graded, i.e. it can be decomposed into a direct sum of subspaces P_i:

$$P = P_0 + P_1 + \cdots + P_n + \cdots$$

such that $P_i \cdot P_j \subset P_{i+j}$, where P_0 is the one-dimensional subspace spanned by the identity of the algebra. An automorphism g of an S-algebra P will be called admissible if $gP_i \subset P_i$ for all i.

The following theorem is due to D. Hilbert (see, for example, (3), p. 91). If G is a reductive group of admissible automorphisms of an S-algebra P, then the algebra $I(P, G)$ is also an S-algebra.

The following generalization of this theorem holds.

Theorem 3. *Let G be a connected Lie group of admissible automorphisms of an S-algebra P; let R be the radical of G. If R is a maximal nilpotent or maximal solvable subgroup of some connected reductive group of admissible automorphisms of P, then the algebra $I(P, G)$ is an S-algebra.*

4. Consider some complex analytic finite-dimensional representation of a semisimple group G:

$$T = k_1 l_1 + k_2 l_2 + \cdots + k_{pl} p,$$

where l_i denotes an irreducible representation with signature* $(l_i^1, \ldots, l_i^r)$; l_i^j are nonnegative integers; r is the rank of G; k_i is the multiplicity with which l_i occurs in T. The vector l_T with coordinates

$$l_T^j = \sum_{s=1}^{p} k_s l_s^j$$

will be called the generalized signature of the representation T (the representation is not determined uniquely by it). Let N_s be the dimension of the representation with signature $(0, \dots, 0, 1, 0, \dots, 0)$ (the one is in the s-th place).

Theorem 4. *Let a representation of a semisimple group G with generalized signature $l = (l^1, \dots, l^r)$ be given in a space V. If there exists an i such that $l^i > N_i$, then the dimension of an orbit of maximal dimension in V is equal to the dimension of the group. Moreover, in V there exists an invariant affine algebraic subvariety M of smaller dimension such that: 1) the dimension of every orbit in $V \setminus M$ is equal to the dimension of the group; 2) the dimension of every orbit in M is less than the dimension of the group.*

Hence, and from the results of the work (4), one can obtain

Corollary. *Let the hypotheses of Theorem 4 be satisfied. Then the number of algebraically independent invariant polynomials on V is equal to the difference between the dimension of V and the dimension of the group G.*

This corollary was previously known only in the simplest special cases.

Remark. Apparently the following conjecture is true. Let G be a group of linear transformations of a finite-dimensional vector space V. If G is irreducible and $\dim G \leq \dim V$, then the dimension of an orbit of maximal dimension is equal to the dimension of the group G.

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* A signature is the collection of coordinates of the highest weight in a basis consisting of generators of the semigroup of highest weights.