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V. B. GOSTEV, A. R. FRENKIN

SINGLE-NUCLEON STATES IN A MODEL WITH A FIXED SOURCE

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In connection with the study of meson-nucleon resonances, it is of interest to consider simple models of field theory with a fixed source. Some features of the meson-nucleon system can be traced in the exactly solvable model considered below with three possible states of the nucleons. This model was proposed in paper ([1]) and studied in paper ([2]). In the present note an exact solution of the Schrödinger equation will be given without using the summation of perturbation-theory series.

In the model four kinds of particles are introduced: fixed fermions A, B, C–they may be regarded as nucleons and an excited state of the nucleon–and a relativistic boson θ. We write the Hamiltonian of the system in the form

$$H = m_{A0}A^+A + m_{B0}B^+B + m_C C^+C + \int d^3k\, \omega a^+(k)a(k) + \\ + \lambda_{01}\int d^3k\, u(\omega)\,[A^+Ba(k) + \text{h.c.}] + \lambda_{02}\int d^3k\, u(\omega)\,[B^+Ca(k) + \text{h.c.}], \tag{1}$$

where $\omega = \omega_k = \sqrt{k^2+\mu^2}$, $u(\omega)$ is a real cutoff function (for a point source $u(\omega) = 1/\sqrt{2\omega}$); $A^+(A)$, $B^+(B)$, $C^+(C)$, and $a^+(k)(a(k))$ are the creation (annihilation) operators of the particles A, B, C, and θ, obeying the Fermi and Bose commutation rules. All quantities with the index 0 are unrenormalized.

The Hamiltonian (1) is a generalization, to the case of three heavy particles, of the Hamiltonian of the Lee model ([3,4]).

We seek the state of a physical A-particle in the form

$$|A\rangle = Z_A^{1/2}\left[A^+ + \int d^3k\, \varphi_1(\omega_k)B^+a^+(k) + \\ + \frac{1}{\sqrt{2!}}\int d^3k\, d^3l\, \varphi_2(\omega_k,\omega_l)C^+a^+(k)a^+(l)\right]|0\rangle, \tag{2}$$

where $|0\rangle$ is the vacuum state; Z_A is a normalization constant (the renormalization constant of the operator A^+), determined from the condition

$$(A|A) = 1. \tag{3}$$

The Schrödinger equation

$$H|A\rangle = m_A|A\rangle \tag{4}$$

is equivalent to the following relations between the wave functions $\varphi_1(\omega_k)$, $\varphi_2(\omega_k, \omega_l)$:

$$m_A = m_{A0} + \lambda_{01} \int d^3k\, u(\omega_k)\varphi_1(\omega_k), \tag{5}$$

$$m_A\varphi_1(\omega_k) = (m_{B0} + \omega_k)\varphi_1(\omega_k) + \lambda_{01}u(\omega_k) +$$

$$+\sqrt{2}\lambda_{02} \int d^3l\, u(\omega_l)\varphi_2(\omega_k, \omega_l), \tag{6}$$

$$m_A\varphi_2(\omega_k, \omega_l) = (m_C + \omega_k + \omega_l)\varphi_2(\omega_k, \omega_l) + \frac{1}{\sqrt{2}}\lambda_{02}[u(\omega_k)\varphi_1(\omega_l) + u(\omega_l)\varphi_1(\omega_k)]. \tag{7}$$

After renormalizations of the mass of the B-particle and of the interaction constant λ_{02}, which coincide with the renormalizations in the Lee model ([4–5]), equations (6) and (7) lead to the integral equation for $\varphi_1(\omega_k)$

$$-h(\omega_0 + b - \omega_k)\varphi_1(\omega_k) = -\lambda_{01}Z_{Bu}(\omega_k) + \gamma u(\omega_k) \int \frac{u(\omega_l)\varphi_1(\omega_l)\, d^3l}{\omega_l + \omega_k - \omega_0 - b}, \tag{8}$$

where $\gamma = \lambda_{02}^2 Z_B = \lambda_2^2$ is the square of the renormalized coupling constant of the $BC\theta$-interaction; $Z_B = 1 - \gamma \int \frac{d^3k u^2(\omega)}{(b-\omega)^2}$ is the renormalization constant of the operator $B^+(B)$; $\omega_0 = m_A - m_B < \mu$; $b = m_B - m_C < \mu$; m_B is the mass of the physical B-particle (the V-particle in the Lee model);

$$h(z) = (z - b)\left[1 + 4\pi(z - b)\gamma \int_\mu^\infty \frac{\sqrt{\omega a'^2 - \mu^2}\, \omega' u^2(\omega')\, d\omega'}{(\omega a'^2 - b^2)(\omega' - z)}\right]; \tag{9}$$

the properties of $h(z)$ have been investigated in [4] (see also [5]). In what follows we restrict ourselves to values $\gamma < \gamma_{\text{crit}}$ $(0 < Z_B \leq 1)$, when $h(z)$ has the single zero $z = b$.

With the substitution

$$\varphi_1(\omega) = \gamma \frac{u(\omega)}{h(\omega_0 + b - \omega)} \psi(\omega) \tag{10}$$

equation (8) is transformed to the form

$$\psi(\omega) = K - \frac{1}{\pi} \int_\mu^\infty \frac{\operatorname{Im} h(\omega')\psi(\omega')\, d\omega'}{h(b + \omega_0 - \omega)(\omega' + \omega - \omega_0 - b)}, \tag{11}$$

where

$$\operatorname{Im} h(\omega) = \operatorname{Im} h(\omega + i\varepsilon) = \theta(\omega - \mu) 4\pi^2 \gamma \sqrt{\omega^2 - \mu^2}\, \omega u^2(\omega), \tag{12}$$

$$K = \lambda_{01} \gamma^{-1} Z_B.$$

An integral equation with such a kernel, but with an inhomogeneous term of the form $-1/(\omega - b)$, decreasing at infinity, for $b = 0$ was solved in [6] by checking an answer found earlier with the aid of dispersion relations [7]. A more consistent method for solving a similar equation with a pole inhomogeneous term is set forth in [2,8,9].

Analogous techniques can also be applied to solving equation (11), despite the fact that $\psi(\omega)$ does not decrease as $|\omega| \to \infty$. To this end we introduce the function

$$\alpha(\omega) = h(\omega)\psi(\omega) + h(\omega_0 + b - \omega)\psi(\omega_0 + b - \omega), \tag{13}$$

which has no cuts on the real axis. From expressions (12) and (13) it follows that $\alpha(\omega)$ has no poles, and it is a polynomial of first degree in ω; however, by virtue of the symmetry $\alpha(\omega) = \alpha(\omega_0 + b - \omega)$, $\alpha(\omega)$ is actually constant:

$$\alpha(\omega) = \alpha(\omega_0) = h(\omega_0)\psi(\omega_0) = \text{const}. \tag{14}$$

Construct the function

$$f(\omega) = (\omega - b)[\psi(\omega_0 + b - \omega) - K]/h(\omega). \tag{15}$$

Having determined $\psi(\omega_0 + b - \omega) - K$ with the aid of equation (11) and using the properties of $h(\omega)$, one can note that $f(\omega)$ has no poles and is analytic in the complex ω-plane with a cut along the real axis from μ to ∞, and that $\lim_{|\omega|\to\infty} f(\omega) = 0$. Therefore, by Cauchy' s theorem,

$$f(\omega) = \frac{1}{\pi}\int_{\mu}^{\infty} \frac{\operatorname{Im} f(\omega')\, d\omega'}{\omega' - \omega}, \tag{16}$$

where

$$\operatorname{Im} f(\omega) = \operatorname{Im} f(\omega + i\varepsilon).$$

For the imaginary part of $f(\omega)$, equalities (13) and (14) give

$$\operatorname{Im} f(\omega) = \frac{h(\omega_0)\psi(\omega_0)(\omega - b)}{h(\omega_0 + b - \omega)} \operatorname{Im} \frac{1}{h(\omega)} - K(\omega - b) \operatorname{Im} \frac{1}{h(\omega)}. \tag{17}$$

Substituting the expressions for $f(\omega)$ and $\operatorname{Im} f(\omega)$ into identity (16), after the change of argument $\omega \to \omega_0 + b - \omega$, we obtain a relation connecting $\psi(\omega)$ and $\psi(\omega_0)$:

$$\psi(\omega) = K + \frac{h(\omega_0 + b - \omega)h(\omega_0)\psi(\omega_0)}{\omega_0 - \omega} A(\omega) - K\frac{h(\omega_0 + b - \omega)}{\omega_0 - \omega} B(\omega), \tag{18}$$

where

$$A(\omega) = \frac{1}{\pi}\int_{\mu}^{\infty} \frac{(\omega' - b)\, d\omega'}{(\omega' + \omega - \omega_0 - b)\, h(\omega_0 + b - \omega')} \operatorname{Im} \frac{1}{h(\omega')}, \tag{19}$$

$$B(\omega) = \frac{1}{\pi}\int_{\mu}^{\infty} \frac{(\omega' - b)\, d\omega'}{(\omega' + \omega - \omega_0 - b)} \operatorname{Im} \frac{1}{h(\omega')}. \tag{20}$$

Eliminating $\psi(\omega_0)$, we find the final expression for $\psi(\omega)$:

$$\psi(\omega) = K\Phi(\omega) = K\left\{1 + \frac{h(\omega_0 + b - \omega)}{\omega_0 - \omega}\left[\frac{h(\omega_0)(1 - B(\omega_0))A(\omega)}{1 - h(\omega_0)A(\omega_0)} - B(\omega)\right]\right\}. \tag{21}$$

The solution (21) was checked by direct substitution into equation (11). Because of its complexity, the verification of the solution is not presented.

The integral (20) is evaluated by contour integration [4]

$$B(\omega) = [B(\omega) - B(\omega_0)] + B(\omega_0) = -1 + \frac{\omega_0 - \omega}{h(\omega_0 + b - \omega)} + B(\omega_0), \tag{22}$$

$$B(\omega_0) = \frac{1}{\pi} \int_{\mu}^{\infty} d\omega' \, \mathrm{Im} \, \frac{1}{h(\omega')} = 1 - Z_B^{-1}, \tag{23}$$

after which the form of $\Phi(\omega)$ is simplified:

$$\Phi(\omega) = Z_B^{-1} \frac{h(\omega_0 + b - \omega)}{\omega_0 - \omega} \frac{1 + h(\omega_0)[A(\omega) - A(\omega_0)]}{1 - h(\omega_0) A(\omega_0)}. \tag{24}$$

The vanishing of the denominator of $\psi(\omega)$,

$$1 - h(\omega_0) A(\omega_0) = 0, \tag{25}$$

corresponds to the appearance of $B-\theta$-bound states when the $AB\theta$-interaction is switched off ($V-\theta$-bound states in the Lee model [8,10]).

Substituting the values of $\varphi_1(\omega)$ from (10), K from (12), and $\psi(\omega)$ from (21), (24) into relation (5), we obtain the characteristic equation

$$m_A - m_{A0} - Z_B \left(\frac{\lambda_{01}}{\lambda_2} \right)^2 \frac{1}{\pi} \int_{\mu}^{\infty} \frac{\mathrm{Im} \, h(\omega') \Phi(\omega') \, d\omega'}{h(\omega_0 + b - \omega')} = 0, \tag{26}$$

which relates the quantity m_A to m_{A0}, γ, λ_{01}^2. The principal consequence of this equation—the possibility, for $0 < Z_B < 1/2$, of the appearance of two one-particle $|A\rangle$ states with the same bare mass m_{A0} and with different signs of $\delta m_A = m_A - m_{A0}$—is in qualitative agreement with the consequence of the approximate characteristic equation of the model. Therefore the exact solution (26) may serve as an a posteriori justification for replacing the kernel of the integral equation (8) by a degenerate one.

A study of the behavior of Z_A, the renormalized coupling constant $\lambda_1 = \lambda_{01} Z_A^{1/2} Z_B^{1/2} Z^{-1}$, and the renormalization constant of the $AB\theta$ vertex Z as functions of γ leads to the conclusion that Z_A tends to zero and λ_1 changes sign discontinuously at γ satisfying equation (25).

In conclusion we note that the wave functions of the continuous spectrum for the scattering of θ-particles by B-particles can be found exactly by solving the integral equation with kernel (11) and an inhomogeneous term of the form $K(\omega_0) = 1/(\omega - b)$, where $K(\omega_0)$ is a linear functional of $\psi(\omega, \omega_0)$. The solution of this problem presents no difficulty, since inhomogeneous equations with a separable term have been solved in (6, 9, 11).

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Moscow State University
named after M. V. Lomonosov

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Note: Figure translations are in progress. See original paper for figures.