Reports of the Academy of Sciences of the USSR

MATHEMATICS

1966

SovietRxiv

Abstract

Full Text

Reports of the Academy of Sciences of the USSR
1966. Volume 167, No. 4

UDC 517.946

MATHEMATICS

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AN ANALYTIC APPROACH TO THE PROBLEM OF EQUATIONS OF THE FIRST KIND

(Presented by Academician M. A. Lavrent' ev on 30 VI 1965)

The paper studies the equation of the first kind

$$Ax = f, \tag{1}$$

where x, f are elements of some Hilbert space H; A is a positive self-adjoint operator with everywhere dense domain of definition $D(A)$, for which zero is a point of the spectrum but is not an eigenvalue. It is proved that if x is a solution of (1) and $\|A\| \leq 1$, then

$$x = \sum_{n=0}^{\infty} (E - A)^n f,$$

where E is the identity operator. If A is an unbounded operator and equation (1) is solvable, then its solution can be found by the formula

$$x = f + \sum_{n=2}^{\infty} \frac{(-1)^{n+1}}{n!} (A - E) \dots (A - (n-1)E) f. \tag{2}$$

Formula (2) is proved under the assumption that $f \in D(A^n)$, $\|A^n f\| \leq C$, $n = 0, 1, 2, \dots$, where C does not depend on n. Formula (2) is equivalent to the method of successive approximations according to the scheme

$$x_n = \frac{1}{n} f + x_{n-1} - \frac{1}{n} A x_{n-1}, \quad n \geq 2, \quad x_1 = f. \tag{3}$$

The general procedure for obtaining various formulas for the solution of equation (1) is as follows. Assuming that $f \in D(A^n)$, $n = 0, 1, 2, \dots$, we construct, analytic in the half-plane $\operatorname{Re} z > 0$, the function $A^z x$, knowing its values $A^n x = A^{n-1} f$

at the points $z = n$, $n \geq 1$. As the solution of equation (1) we take $\lim_{z \to +0} A^z x$. The function $A^z x$ is determined uniquely and depends only on f. If, for the constructive determination of $A^z x$, one uses, for example, the formulas of paper (3), p. 149, then we obtain

$$x = \lim_{z \to +0} x_z, \tag{4}$$

where $zx_z + Ax_z = f$, $z > 0$. Formulas (2)-(4) are obtained solely on the basis of the fact of existence of a solution. If one assumes in advance that the sought solution is "sourcewise representable," i.e. $x = Ay$, $y \in H$, then, for example, in formula (4) one can obtain an estimate for the rate of approximation of the form $\|x - x_z\| \leq z\|y\|$.

It seems useful to relate certain questions of the general theory of equations of the second kind to the theory of equations of the first kind.

For example, consider the equation

$$x - \varepsilon A x = f, \tag{5}$$

where $A = A^*$, $A > 0$, $\varepsilon = \varepsilon_0 + i\delta$, $\delta > 0$, δ small. It is shown that if $f \in D(A^n)$, $\|A^n f\| \leq C$, $n = 0, 1, 2, ...$, where C does not depend on n, then the solution

equation (5) is found by the formula

$$x = f + \sum_{n=1}^{\infty} \rho_n(\varepsilon) \varphi_n(A) f + r_N, \tag{6}$$

where $\varphi_n(A) = A(A - iE) \ldots (A - i(n-1)E)$, $\rho_n(\varepsilon) = \varepsilon^n / (1 - i\varepsilon) \ldots (1 - in\varepsilon)$, $\|r_N\| \leq K(\alpha)/\delta^N \alpha$, $\alpha = \delta/(\varepsilon_0^2 + \delta^2)$, $K(\alpha)$ is a certain constant, $K(0) > 0$, and N is arbitrary.

The behavior of the solution of equation (5) as $\varepsilon \to 0$ is connected with the study of equation (1) by means of formulas (4), if A^{-1} exists. Let us consider the question of the asymptotics of the solution of equation (5) as $\varepsilon \to 0$. Suppose that, for some $\alpha > 0$, $\beta > 0$, integer $m \geq 1$, and $|\varepsilon| \leq 1$, the following conditions are satisfied: $\delta \to 0$, $\alpha|\varepsilon_0|^{m+1} < \delta < \beta|\varepsilon_0|$. If, moreover, $f \in D(A^n)$, $\|A^n f\| \leq C$, $n = 0, 1, 2, ...$, then for any N and for some $\gamma = \gamma(\alpha, \beta, m)$ we have

$$\left\| x - \sum_{n=0}^{N} \varepsilon^n A^n f \right\| \leq \gamma C |\varepsilon|^{N+1}. \tag{7}$$

Estimate (7) is obtained from the identity

$$x = \sum_{n=0}^{N} \varepsilon^n A^n f + \varepsilon^{N+1} \left\{ A^{N+1} \left(\sum_{n=0}^{m-1} \varepsilon^n A^n f \right) + \varepsilon^m (E - \varepsilon A)^{-1} A^{m+N+1} f \right\}$$

and the known estimate

$$\|\varepsilon(E - \varepsilon A)^{-1} g\| \le \|g\| \left| \operatorname{Im} \frac{1}{\varepsilon} \right|, \qquad g \in H.$$

From inequality (7) it follows that

$$x \sim \sum_{n=0}^{\infty} \varepsilon^n A^n f.$$

Let us consider the question of the asymptotics of the solution of equation (5) obtained by the "limiting absorption principle." Suppose there exists an element $x(\varepsilon_0) \in D(A)$ such that $x(\varepsilon_0) - x \to 0$ as $\delta \to 0$, where x satisfies (5). Suppose, in addition, that $f \in D(A^n)$, $\|A^n f\| \le C$, and $\|(E - \varepsilon A)^{-1} A^n f\| \le C\varepsilon_0^{-m}$, where m is some integer, $m \ge 0$, C does not depend on n, $n = 0, 1, 2, ...$ Then

$$x(\varepsilon_0) \approx \sum_{n=0}^{\infty} \varepsilon_0^n A^n f.$$

The method used to derive estimate (7) was applied to obtain the asymptotics of the solution of the equation $\varepsilon \Delta u - B(u) = f$, which arises in the theory of diffraction of waves on unbounded surfaces in three-dimensional space, where $B(u) = 2i(\nabla\mu, \nabla u) + iu\Delta\mu$, $(\nabla\mu)^2 = 1$, $\varepsilon = \varepsilon_0 - i\varepsilon_0^m$, $m \ge 1$, $\varepsilon_0 \to 0$.

Let us consider some equations of the first kind that arise in mathematical physics. It is shown that the solution of many such equations reduces to the solution of the ill-posed problem of constructing a function $F(z)$, analytic for $\operatorname{Re} z > 0$ and continuous for $\operatorname{Re} z \ge 0$, from the known numbers $F(n)$, $n = 1, 2, ...$ This problem has a unique solution if, for some $M > 0$, $B > 0$, $0 < \varepsilon < \pi$, the conditions $|F| \le M \exp(B|z|)$, $|F(i|z|)| \le M \exp((\pi - \varepsilon)|z|)$ are satisfied. (See (3), p. 144.) It is shown that, under these assumptions, for all z from any bounded domain of the right half-plane the estimate $|F(z)| \le K = K(M, B, \varepsilon, m)$ holds, where $m = \max |F(n)|$, $K \to 0$ as $m \to 0$ and fixed M, B, ε. In this sense the problem of constructing $F(z)$ is well-posed if the numbers $F(n)$, M, B, ε, $n = 0, 1, ...$, are given. With the help of the formulas from (3), p. 152, it is proved that, under natural restrictions on $F(n)$, the preceding problem leads to the singular equation

$$g(t) - \frac{1}{\pi i} \int_{-\infty}^{+\infty} \frac{g(\xi)\, d\xi}{\xi - t} = H(t),$$

where

$$g(t) = \pi t F(it)/\operatorname{sh}\pi t, \qquad H(t) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} n F(n)}{n - it}.$$

Let us consider some equations of the first kind arising in potential theory. Consider the inverse problem of the logarithmic potential, which reduces to determining the boundary L of a domain D, if for large $|z|$ the function $U(z)$ is known (the exterior complex potential)

$$U(z) = \iint_D \frac{d\xi\, d\eta}{z - \zeta}, \tag{8}$$

where $z = x + iy$, $\zeta = \xi + i\eta$. The integral (8) for $z \in D$ is called the interior complex potential. Let the boundary L in polar coordinates (r, θ) be given by the equation $r = \varphi(\theta)$, $|\theta| \leq \pi$. We shall show that if $\varphi(\theta)$ is known for $\pi - |\theta| \leq \varepsilon$, then the nonlinear equation (8) for $\varphi(\theta)$ is solved by methods of analytic continuation. Let us explain what has been said. Let $|\theta| \leq \pi$, $m = \min \varphi(\theta)$, $M = \max \varphi(\theta)$. Then for $|z| < m$

$$U(z) = \pi \bar{z} - \sum_{n=0}^{\infty} a_{-n} z^n$$

and for $|z| > M$

$$U(z) = \sum_{n=0}^{\infty} a_n z^{-n-1},$$

where $a_{-n} = f(-n-1)$ for $n = 0, 2, 3, ...$; $a_{-1} = f(-2) + \pi$; $a_n = f(n)$, $n = 0, 1, 2, ...$; $f(\pm n) = f(z)$ for $z = \pm n$;

$$f(z) = \frac{1}{z+2} \int_{|\theta| \leq \pi} \varphi^{2+z}(\theta) e^{iz\theta}\, d\theta. \tag{9}$$

Since a_n and $\varphi(\theta)$ for $\pi - |\theta| \leq \varepsilon$ are given, one can construct the entire function

$$F(z) = \int_{|\theta| \leq \pi - \varepsilon} \varphi^{2+\varepsilon}(\theta) e^{iz\theta}\, d\theta,$$

having first determined $F(n)$, $n = 1, 2, ...$, and from $F(z)$ determine $f(z)$ and the numbers $a_{\pm n}$, i.e. one can construct the interior and exterior complex potentials of the domain D. The boundary L is then determined as the set of those points z where the two potentials coincide. In an analogous way, also in the spatial case of the inverse problem of potential theory, knowledge of part of the boundary makes it possible to determine the remaining part of the boundary by methods of interpolation of entire functions.

The problem of finding a real function $\rho(\theta)$ has been considered and solved, if the numbers

$$\int_{|\theta|\le\pi} \rho(\theta)\Phi^n(\theta)\,d\theta = c_n, \qquad n = 0, 1, 2, \dots, \tag{10}$$

are given, where $\Phi(\theta) = e^{i\theta}\varphi(\theta)$ is a given function; $\varphi(\theta) > 0$; $\varphi, \varphi', \varphi''$ are 2π-periodic functions; $\sqrt[n]{|c_n|} < \min\varphi(\theta)$. A spatial analogue of problem (10) has been studied—the problem of finding a function $\rho(\lambda, \theta)$ from the system of equations

$$\int_{\omega} \rho\Phi^{n+3} P_n(\cos\gamma)\,d\omega = A_n, \qquad n = 0, 1, 2, \dots, \tag{11}$$

where ω is the unit sphere; P_n is a Legendre polynomial; $\Phi = \Phi(\lambda, \theta)$; γ is the angle between the point of the sphere (λ, θ) and the vector $(\Phi(\lambda', \theta'), \lambda', \theta')$; λ', θ' are variables of longitude and latitude on the sphere; $d\omega = \sin\theta'\,d\lambda'\,d\theta'$; $A_n = A_n(\lambda, \theta)$–

given spherical functions;

$$\sqrt[n]{|A_n|} < \min_{\omega} \Phi(\lambda, \theta); \qquad |\lambda| \le \pi; \qquad 0 \le \theta \le \pi;$$

Φ is a function prescribed on ω, sufficiently smooth.

On the basis of problem (11), new equations have been obtained for the solution of the inverse problem of the Newtonian potential in space.

Let us consider one problem from the theory of difference schemes that leads to the inverse problem of the logarithmic potential. Let $u(s, n)$, $s = 0, \pm1, \pm2, \dots$, satisfy the difference equation

$$u(s, n+1) = \sum_{r=-\infty}^{+\infty} c_r u(s + r, n). \tag{12}$$

We pose the problem of finding the numbers c_r. Let $c_r = c_{-r}$ and

$$\varphi(\theta) = \sum c_r e^{ir\theta}$$

be a continuous function, $\varphi(\theta) > 0$, $|\theta| \le \pi$. Let $u^{(m)}(s, n)$ be a family of solutions of equation (12), with $u^{(m)}(s, 0) = 0$ for $s \ne m$ and $u^{(m)}(m, 0) = 1$.

Suppose the numbers $b_m = u^{(m)}(2, m)$, $m = 2, 3, ...$, are given. Then, for finding $\varphi(\theta)$, we have the equations

$$\int_{|\theta|\leq\pi} \varphi^m(\theta) e^{i(m-2)\theta}\, d\theta = 2\pi \bar{b}_m, \qquad m = 2, 3, ... \tag{13}$$

It is not difficult to see that, knowing $\bar{b}_m$, we know the exterior complex potential of the domain $D = \{z : \ |z| \leq \varphi(\arg z)\}$. Formula (13) is obtained from the equality

$$u(s, n) = \sum_{m=0}^{\infty} \frac{1}{2\pi} u(m, 0) \int_{|\theta|\leq\pi} \varphi^n(\theta) e^{i(s-m)\theta}\, d\theta$$

(see (1), p. 133). Differential analogues of problem (12)–(13) were studied in paper (2).

The question of the correctness of the Cauchy problem is considered for elliptic systems in the sense of Petrovskii of arbitrary order, in which the coefficients of the highest derivatives are constant, while the coefficients of the lower derivatives are entire functions. The investigation is based on a formula which, in particular, has the following form.

If the function $u(x, y, z)$ is harmonic in a bounded domain D of 3-dimensional space and on a part Γ_1 of the boundary Γ the value $u|_{\Gamma_1} = \varphi$ and the normal derivative $u_n|_{\Gamma_1} = \psi$ are known, then for $P = (x, y, z) \in D$

$$u = v - v^*, \tag{14}$$

where

$$v = \frac{1}{4\pi} \int_{\Gamma_1} \varphi \frac{\partial}{\partial n} \frac{1}{|P - Q|}\, d\sigma_Q - \frac{1}{4\pi} \int_{\Gamma_1} \psi \frac{1}{|P - Q|}\, d\sigma_Q,$$

and v^* is the harmonic continuation of v into D through Γ_1. Formula (14) is also generalized to equations of parabolic type.

Received
26 VI 1965

CITED LITERATURE

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