REDUCTION OF A PLANE GRAPH TO AN EDGE BY STAR-TRIANGLE TRANSFORMATIONS

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Abstract

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REDUCTION OF A PLANE GRAPH TO AN EDGE BY STAR–TRIANGLE TRANSFORMATIONS

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1. Let us consider the following transformations of undirected multigraphs*.

(I) Replacement of two edges joining one and the same pair of vertices by a single edge. (II) Replacement of a chain of two edges joining a pair of vertices by a single edge joining these same vertices. (III) Replacement of three edges forming a triangle with vertices a, b, c by a star: three edges $[a, d]$, $[b, d]$, $[c, d]$, with a new vertex d. (IV) The transformation inverse to (III).

Eikers ([2]), and then Lehman ([3]), conjectured that every plane multigraph with two poles (distinguished vertices), satisfying certain restrictions natural for transportation networks, can be reduced to the single edge joining these poles by means of a finite number of transformations of types (I)–(IV) preserving the poles. In the present note we set forth the scheme of proof of the following lemma, from which the validity of this conjecture follows.

Main lemma. *Let G be a finite connected plane undirected multigraph possessing the following properties: G has no more than two poles and at least one vertex that is not a pole; G has no loops and no vertices of degree 1 distinct from the poles. Then there exists a finite (possibly empty) sequence of transformations of types (III), (IV), preserving the poles and the property of multigraphs of being plane, which transforms G into a plane multigraph G' such that it possesses all the listed properties of the multigraph G and at least one of the transformations (I), (II), preserving the poles, is applicable to it.*

The following is easily obtained from the main lemma.

Theorem. *Let G be a plane undirected multigraph with two poles. In order that there exist a finite (possibly empty) sequence of transformations (I)–(IV), preserving the poles, reducing G to the single edge joining these poles, it is sufficient that the multigraph G be finite, have no isolated vertices, and that through each of its edges there pass an elementary chain**, joining the poles.*

From this follows the validity of the conjecture indicated above.

2. The proof of the main lemma (items 2–12) will be carried out by contradiction. Let Γ be the family of all multigraphs satisfying the conditions of the theorem, for which nevertheless its conclusion is false and which moreover have the least possible number of edges. Suppose that Π is nonempty. Then all elements of Γ are graphs in which every vertex distinct from a pole has degree > 2; transformations (III), (IV) do not lead out of Γ.

3. Considering an arbitrary graph $G \in \Gamma$, given by some realization on the sphere Σ, we introduce the notion of a completely admissible set—

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* In terminology we follow ([1]).
** That is, a chain without coincident vertices.

(with respect to the order of the transformations (III), (IV)), and also a number of auxiliary concepts.

Let $\mu = (v_1, v_2, \ldots, v_p)$, $v_i = [b_{i-1}, b_i]$, $1 \leq i \leq p$, be an arbitrary (oriented) chain in the graph G. The angle from v_i to v_{i+1}, $1 \leq i < s$, counted counterclockwise (respectively clockwise), will be called the right (respectively left) angle of the chain μ at the vertex b_i; the vertex b_i, $1 \leq i < s$, will be called a right (left) vertex of the chain μ, if inside the right (left) angle of the chain μ at the vertex b_i there are no edges of the graph G incident with this vertex. Let $C = (u_1, u_2, \ldots, u_s)$, $u_i = [a_{i-1}, a_i]$, $1 \leq i \leq s$, be such a simple cycle* in the graph G that, if it has coincident vertices, then at them C does not cross itself, but only touches itself; suppose, moreover, that every such "multiple" vertex is either a right or a left vertex of the cycle C; then the cycle C will be called a regular cycle. Obviously, a regular cycle C, considered as a subset of the sphere Σ, determines two closed subsets T_1 and T_2 of it with common boundary C, which have the following property: if a sufficiently small neighborhood of any interior point of an arbitrary edge u_i, $1 \leq i \leq s$, of the cycle C is taken, then all points of this neighborhood lying to the left of the edge u_i (in the direction of traversal of the cycle C) are contained in the set T_1, and all points lying to the right are contained in the set T_2. Such closed sets T_1 and T_2 will be called, respectively, the left and right sets determined by the cycle C.

Let T be either the left or the right set determined by the cycle C. A vertex of the cycle C which is not a pole will be called principal if every edge of the graph G incident with this vertex is contained in T. Suppose a certain partition is given of all non-principal vertices of C into two disjoint classes: special vertices and inessential vertices; here the special vertices are chosen from among those vertices for which there exists exactly one incident edge of the graph G lying outside T. An edge of the cycle C will be called special (for the given partition) if both its ends are inessential vertices. The peculiarities of the cycle C will mean both its special vertices and its special edges. If the number of peculiarities of the cycle C under the given partition is < 3, then such a partition will be called admissible. If, moreover, the poles of the graph G are neither inside T nor among the principal or special vertices of C, then such a partition will be called completely admissible. The left or right set T determined by the cycle C will be called an admissible (completely admissible) set if for the cycle C there exists an admissible (completely admissible) partition.

4. We shall show that every graph $G \in \Gamma$ can be realized on the sphere Σ in such a way that there is a regular cycle C of the graph G determining a completely admissible set. First we introduce one auxiliary concept. Let $\mu = (v_1, v_2, \dots, v_p)$, $v_i = [b_{i-1}, b_i]$, $1 \le i \le p$, be such a chain of the graph G, given by some realization on the sphere Σ, that all its vertices except the initial and terminal ones are, alternately (in the order of occurrence in μ), right and left vertices. Then the chain μ will be called a perfect chain.

For the proof it is evidently sufficient to consider the case when G has two poles P_1 and P_2. Suppose that at least one of the poles, for example P_1, has degree > 1. We then construct such a perfect chain

$$\mu_1 = (v_1, v_2, \dots, v_p), \quad v_i = [b_{i-1}, b_i], \quad 1 \le i \le p,$$

that: a) $b_0 = P_1$; b) $(B_p \equiv P_1) \vee (B_p \equiv P_2) \vee (\exists i_0 (1 \le i_0 < p - 1 \ \&\ v_p \equiv v_{i_0}))$; c) the chain μ_1 is minimal (with respect to inclusion) among all perfect chains possessing properties a) and b). If $b_p \equiv P_1$, but $v_p \not\equiv v_1$, then μ_1 is a regular cycle and determines two admissible sets, of which one (not containing P_2) is completely admissible. In the contrary case we construct

* That is, a cycle without coincident edges.

such a perfect chain $\mu_2 = (w_1, w_2, \dots, w_q)$, $w_j = [c_{j-1}, c_j]$, $1 \le j \le q$, that: a) $c_0 \equiv P_1$; b) $w_1 \ne v_1$; c) $(c_q \equiv P_1) \vee (c_q \equiv P_2) \vee (\exists j_0 (1 \le j_0 < q - 1 \ \&\ w_q \equiv w_{j_0})) \vee (\exists i_0 (1 \le i_0 \le p \ \&\ w_q \equiv v_{i_0}))$; d) if $p > 1$ and $q > 1$, then the vertices b_1 and c_1 are simultaneously either right or left; e) the chain μ_2 is minimal (with respect to inclusion) among all perfect chains possessing properties a)–d). It is not difficult to show that at least one of the maximal regular cycles contained either in the chain μ_1, or in the chain μ_2, or in the union of the chains μ_1 and $\mu_2^{-1} = (w_q, w_{q-1}, \dots, w_1)$, determines a completely admissible set.

If both P_1 and P_2 have degree 1, then for the graph G_1, obtained from G by deleting P_1 and the edge $[P_1, P_1']$ (where P_1' is the vertex adjacent to P_1), we find a completely admissible set T, regarding P_1' and P_2 as poles of G_1; it remains to return P_1 and $[P_1, P_1']$ in such a way that they lie outside T.

5. We shall call the **weight** of a completely admissible set T the ordered pair (k, l), where k is the number of edges of the graph G contained in T (including the boundary of T), and l is the number of edges lying on the boundary of T. We shall assume that

$$(k, l) < (k', l') \iff (k < k' \vee (k = k' \ \&\ l < l')).$$

For every graph in Γ consider the set of all its possible realizations on the sphere Σ and choose a graph $G_0 \in \Gamma$ such that in some realization of it there exists a completely admissible set T_0 with the least possible weight (among all realizations of all graphs in Γ).

6. Assuming that on the cycle C_0, which determines the thus chosen T_0, the corresponding completely admissible partition is specified, we indicate some properties of C_0 and T_0. a) Every principal vertex of the cycle C_0 has degree ≥ 3. b) No special vertex of C_0 having degree 3 can be adjacent to two inessential vertices of C_0. c) In the graph G_0 there is no such "empty" triangle* that one of its edges lies inside T_0, and the other two lie on C_0, moreover the vertex common to these last two edges is an inessential vertex of C_0. d) If the interior of T_0 is homeomorphic to an open disk, then inside T_0 there is an edge of the graph G_0. e) If the interior of T_0 is not homeomorphic to an open disk, then among the cycles contained in C_0 there is at least one cycle C_0^* having no more than one special feature of the cycle C_0, such that in it the order of immediate succession of edges coincides with the order of immediate succession of these same edges in the cycle C_0, except for one pair of immediately following edges of C_0^*; moreover the cycle C_0^* contains no more than one special feature of the cycle C_0. Let T_0^* be the left or right set determined by the cycle C_0^* and contained in T_0. Then inside T_0^* there is an edge of the graph G_0.

7. We outline the proof of the following assertion. Let $\widehat{T}$ be any such left or right set determined by a regular cycle

$$\widehat{C} = (\hat{u}_1, \hat{u}_2, \dots, \hat{u}_s), \quad \hat{u}_i = [\hat{a}_{i-1}, \hat{a}_i], \quad 1 \leq i \leq s$$

(in the graph G_0), with some (not necessarily admissible) partition specified on $\widehat{C}$, such that $\widehat{T}$, $\widehat{C}$, and the given partition satisfy the following conditions: a) the interior of $\widehat{T}$ is homeomorphic to an open disk; b) there is an r, $1 \leq r \leq s$, such that the chain

$$\hat{\mu} = (\hat{u}_1, \hat{u}_2, \dots, \hat{u}_r)$$

has the following properties: all vertices of the chain $\hat{\mu}$ are inessential; for the vertices $\hat{a}_0$ and $\hat{a}_r$, and only for these vertices of the chain $\hat{\mu}$, there exist edges of the graph G_0 incident with these vertices and lying inside $\widehat{T}$; c) $\widehat{C}$ has ≤ 1 special vertices; d) among the edges $\hat{u}_i$, $r < i \leq s$, there are ≤ 1 special edges of the cycle $\widehat{C}$; if such a special edge exists, then $\widehat{C}$ has no special vertices; e) all vertices and edges of the cycle $\widehat{C}$ not lying on the chain $\hat{\mu}$ have properties a)–c) indicated in item 6 for the vertices and edges of the cycle C_0; f) inside $\widehat{T}$ there are no poles of the graph G_0. Then there exists a completely admissible set contained in $\widehat{T}$ and not coinciding with $\widehat{T}$.

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* That is, one of the regions into which this triangle divides the sphere Σ does not contain edges of the graph G_0.

8. The proof of the assertion of item 7 is carried out by contradiction (items 8–11). Suppose that there exist sets satisfying all the conditions of item 7 for which, nevertheless, the conclusion of item 7 is false. Choose from among them such a set $\hat{T}_0$ that contains the minimal number of edges of the graph G_0. For definiteness one may assume that $\hat{T}_0$ is a left set, determined by the cycle $\hat{C}_0 = (u_1, u_2, \dots, u_s)$, $u_i = [a_{i-1}, a_i]$, $1 \leq i \leq s$.

9. Construct a chain $\mu^* = (v_1, v_2, \dots, v_p)$, $v_j = [b_{j-1}, b_j]$, $1 \leq j \leq p$, satisfying the following conditions: a) $b_0 \equiv a_0$, and this vertex is the left vertex of the chain (u_s, v_1); b) all edges v_j, $1 \leq j \leq p$, lie inside $\hat{T}_0$; c) if, for $1 < j < p$, the vertex b_j lies inside $\hat{T}_0$, then it is a left vertex of the chain μ^*; d) if, for $1 < j < p$, the vertex b_j lies on $\hat{C}_0$, then it is a principal vertex of the cycle $\hat{C}_0$; moreover, there exist exactly two edges of the graph G_0 incident with b_j and lying inside the left angle of the chain μ^* at the vertex b_j; e) for $1 \leq j < p$ there do not exist vertices b_{j-1} and b_j both lying on $\hat{C}_0$; f) the chain μ^* is maximal among all chains satisfying conditions a)–e).

10. Let b_{j_1}, b_{j_2}, $0 \leq j_1 < j_2 \leq p$, be two "neighboring" vertices of the chain μ^* lying on $\hat{C}_0$; $b_{j_1} \equiv a_{i_2}$, $b_{j_2} \equiv a_{i_1}$, $0 \leq i_1 < i_2 \leq s$. Using the minimality of $\hat{T}_0$, one can show that inside the right set determined by the cycle $C_{i_1} = (v_{j_2}, v_{j_2-1}, \dots, v_{j_1+1}, u_{i_2}, u_{i_2-1}, \dots, u_{i_1+1})$ there are no edges of the graph G_0.

11. Using the minimality of $\hat{T}_0$ and item 10, one can show that in the left set T', determined by the cycle $C' = (u_1, u_2, \dots, u_{i_0}, v_p, v_{p-1}, \dots, v_1)$ $(a_{i_0} \equiv b_p)$, there is contained a fully admissible set, which contradicts the assumption about $\hat{T}_0$.

12. If the interior T_0 (item 5) is homeomorphic to an open disk, then, using item 6, it is easy to show that T_0 satisfies all the conditions of item 7 for the set $\hat{T}$; if it is not homeomorphic, these conditions are satisfied by the set T_0^* (item 6). But then the conclusion of item 7, respectively for T_0 or T_0^*, contradicts the assumption of the minimality of the weight of T_0.

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Note: Figure translations are in progress. See original paper for figures.