EXACT THEORY OF STEADY WAVES ON THE FREE SURFACE AND ON THE INTERFACE OF TWO LIQUIDS

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Abstract

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EXACT THEORY OF STEADY WAVES ON THE FREE SURFACE AND ON THE INTERFACE OF TWO LIQUIDS

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In the present article we consider the problem of the existence of steady waves of finite amplitude on the surface of a layer of light liquid of depth h, superposed on the surface of a heavy liquid at rest at infinite depth; we also investigate waves on the interface.

We study irrotational motion in which the interface l_2 and the line l_1 of the free surface are curves having period λ and moving without change of form with some common horizontal velocity c. In order to deal with steady motion, we introduce a moving system of axes OXY through one of the crests D_2 of the curve l_2. We place the origin O at the mean level of the interface of the liquids. We direct the OX axis horizontally in the direction opposite to the direction of the absolute displacement of the line l_2. We direct the OY axis vertically downward. Introduce the complex variables $z_k = x_k + iy_k$ (the index $k = 1$ will characterize the upper liquid and $k = 2$ the lower one) and the corresponding complex potentials

$$f_k = \varphi_k + i\psi_k. \tag{1}$$

We shall assume that at the point D_2, $\varphi_k = \psi_k = 0$. The complex velocities are represented by the formulas

$$W_k = u_k - iv_k = df_k/dz_k. \tag{2}$$

For definiteness we shall assume that $\psi_k > 0$ and, consequently, increase together with y. The kinematic conditions characterizing the motion under consideration are expressed by the formulas:

$$\text{on the line } l_2,\ \psi_1 = \psi_2 = 0; \quad \text{on the line } l_1,\ \psi_1 = q\,(q < 0). \tag{3}$$

At infinity, i.e., when y and ψ_2 simultaneously tend to infinity,

$$W_2 \to c. \tag{4}$$

The dynamic conditions of constant pressure on the free surface l_1 and of continuity of pressure on the interface l_2 have the form

$$gy_1 - \tfrac{1}{2}|W_1|^2 = \text{const} \quad \text{on the line } l_1;$$

$$g\rho_1 y_1 - \tfrac{1}{2}\rho_1 |W_1|^2 = g\rho_2 y_2 - \tfrac{1}{2}\rho_2 |W_2|^2 + \text{const} \quad \text{on the line } l_2. \tag{5}$$

The period λ in the z_k-planes corresponds to the periods λ_k in the f_k-planes. Introduce new independent variables $\zeta_k = e^{2\pi i f_k/\lambda_k} = R_k e^{i\sigma_k}$ and new functions $\omega_k(\zeta_k)$, $W_k = c_k e^{-i\omega_k}$, $\omega_k = \vartheta_k(R_k, \sigma_k) + i\tau_k(R_k, \sigma_k)$, where the positive constant c_1 is chosen in such a way that the Laurent expansion of ω_1 contains no imaginary part in the constant term; c_2 is taken equal to c, owing to which ω_2 will have no constant term in its Maclaurin expansion.

It can be shown that the solution of the problem reduces to determining the function $\omega_1(\zeta_1)$, holomorphic in the annulus Σ_1 $(1 \le |\zeta_1| \le r)$, and the function $\omega_2(\zeta_2)$, holomorphic inside the circle Σ_2 $(|\zeta_2| \le 1)$, satisfying the following conditions. On Γ_2 $(|\zeta_k| = 1)$, i.e., on the interface, there will be ⋯

this condition

$$e^{-\tau_2(\sigma_2)}\, d\sigma_2 = e^{-\tau_1(\sigma_1)} \left\{ \frac{1}{2\pi} \int_0^{2\pi} e^{\tau_2(\sigma_2) - \tau_1(\sigma_1)}\, d\sigma_1 \right\}^{-1} d\sigma_1, \tag{6^1}$$

$$\vartheta_1(\sigma_1) = \vartheta_2(\sigma_2), \tag{6^2}$$

$$(c^2/c_1^2) m_2 e^{2\tau_2(\sigma_2)} \partial\tau_2(\sigma_2)/\partial\sigma_1 - m_1 e^{2\tau_1(\sigma_1)} \partial\tau_1/\partial\sigma_1 = p e^{-\tau_1(\sigma_1)} \sin\vartheta_1(\sigma_1) \tag{6^3}$$

and on $\Gamma_1 (|\zeta_1| = r)$, i.e., on the free surface,

$$\partial\tau_1(r, \sigma_1)/\partial\sigma_1 = p e^{-3\tau_1(r,\sigma_1)} \sin\vartheta_1(r, \sigma_1), \tag{7}$$

where

$$\vartheta_k(\sigma_k) + i\tau_k(\sigma_k) \equiv \omega_k(e^{i\sigma_k}), \qquad m_1 = \rho_1/(\rho_2 - \rho_1), \qquad m_2 = \rho_2/(\rho_2 - \rho_1),$$

$$p = g\lambda_1/2\pi c_1^3, \qquad r = e^{-2\pi q/\lambda_1}.$$

One may regard r, c, c_1 as given; in particular, it may be that $c_1 = c$; then, knowing the functions ω_1, ω_2 and the constant p, q, h and λ_k, λ are determined by the formulas

$$r = e^{-2\pi q/\lambda_1}, \qquad \lambda = \frac{\lambda_k}{2\pi c_k}\int_0^{2\pi} e^{-\tau_k}\cos\vartheta_k\, d\sigma_k, \qquad p = \frac{g\lambda_1}{2\pi c_1^3}, \qquad \int_0^{\lambda}(y_{l_1}+h)\, dx_{l_1} = 0. \tag{8}$$

Moreover, the ordinate a of the vertex D_2 of the line l_2 is determined by the formula

$$\int_0^{\lambda} y_{l_2}\, d\bar{x} = 0. \tag{9}$$

We shall seek the solution of the problem in the form of series arranged in powers of a certain parameter μ. Suppose that the functions ω_k and the constant p can be expanded in series in powers of μ, and write

$$\omega_k(\zeta) = \sum_{n=1}^{\infty}\mu^n\omega_k^{(n)}(\zeta) = \sum_{n=1}^{\infty}\mu^n\left[\vartheta_k^{(n)}(R,\sigma) + i\tau_k^{(n)}(R,\sigma)\right], \tag{10}$$

$$p = \frac{g\lambda_1}{2\pi c_1^3} = \beta + k, \qquad k = \sum_{n=1}^{\infty}k^{(n)}\mu^n. \tag{11}$$

Moreover, assume that

$$\sigma_2 = \sigma_1 + s(\sigma_1), \qquad s(\sigma_1) = \sum_{n=1}^{\infty}\mu^n s^{(n)}(\sigma_1), \tag{12}$$

where $\sigma_1 = \sigma_2 = 0$ corresponds to the vertex D_2 of the line l_2. Substituting the expansions (10), (11), (12) into the conditions (6) and (7), and equating on both sides of the resulting equations terms of the same order with respect to μ, we obtain systems of equations from which all unknown functions and constants can be determined.

The first-order terms give a system of homogeneous equations for determining $\omega_1^{(1)}, \omega_2^{(1)}, \beta$, corresponding to an infinitesimal solution of the problem. The case $c_1 = c$ is considered, in which the solution is found in two different cases:

a) when $\beta = 1$,

$$\omega_1^{(1)}(\zeta) = \omega_2^{(1)}(\zeta) = -i\zeta; \tag{13}$$

b) when $\beta = (\alpha r + r^{-1})/(r - r^{-1})$, $\alpha = m_2 - m_1$,

$$\omega_1^{(1)}(\zeta) = i\{(1+\beta)r^{-1}\zeta - r(1-\beta)\zeta^{-1}\},$$

$$\omega_2^{(1)}(\zeta) = i\{(1+\beta)r^{-1} + (1-\beta)r\}\zeta. \tag{14}$$

By the principle of mathematical induction one can prove that the solution of the problem will have the form

$$\omega_1^{(j)} = i\sum_{\nu=1}^{j}\left(a_{\nu 1}^{(j)}\zeta^{\nu} + b_{\nu 1}^{(j)}\zeta^{-\nu}\right), \qquad \omega_2^{(j)} = i\sum_{\nu=1}^{j} a_{\nu 2}^{(j)}\zeta^{\nu},$$

$$s^j(\sigma_1) = \sum_{\nu=1}^{j} c_\nu^{(j)} \sin\nu\sigma_1, \qquad k = \sum_{\nu=1}^{\infty} k^{(2\nu)}\mu^{2\nu}. \tag{15}$$

In fact, if we assume that we have found $\omega_k^{(1)}, \omega_k^{(2)}, \dots, \omega_k^{(n-1)}$ $(k = 1, 2)$, $s^{(1)}, s^{(2)}, \dots, s^{(n-2)}$ and $k^{(1)}, k^{(2)}, \dots, k^{(n-2)}$ $(k^{(1)} = k^{(3)} = \dots = k^{2m+1} = 0,\ 2m+1 \le n-2)$ in the form (15), then from condition (6^1) it can be found that $s^{(n-1)}$ will have the same form as $s^{(n-2)}$. Next, if we assume that $\omega_1^{(n)}$ and $\omega_2^{(n)}$ $(n > 1)$ have the general form

$$\omega_1^{(n)} = a_0^{(n)} + \sum_{\nu=1}^{\infty}\{(\alpha_{\nu 1}^{(n)} + i a_{\nu 1}^{(n)})\xi^{\nu} + (\beta_{\nu 1}^{(n)} + i b_{\nu 1}^{(n)})\xi^{-\nu}\},$$

$$\omega_2^{(n)} = \sum_{\nu=1}^{\infty}(\alpha_{\nu 2}^{(n)} + i a_{\nu 2}^{(n)})\xi^{\nu}, \tag{16}$$

then the various coefficients can be determined by using conditions (6^1), (6^2), and (7). They give the conditions

$$\vartheta_1^{(n)} - \vartheta_2^{(n)} = \sum_{\nu=1}^{n} A_\nu^{(n)} \sin\nu\sigma_1,$$

$$m_2\tau_2^{(n)\prime} - m_1\tau_1^{(n)\prime} - \beta\vartheta_1^{(n)} - k^{(n-1)}\vartheta_1^{(1)} = \sum_{\nu=1}^{n} B_\nu^{(n)} \sin\nu\sigma_1, \tag{17}$$

$$\tau_1^{(n)\prime}(r,\sigma_1) - \beta\vartheta_1^{(n)}(r,\sigma_1) - k^{(n-1)}\vartheta_1^{(1)}(r,\sigma_1) = \sum_{\nu=1}^{n} C_\nu^{(n)} \sin\nu\sigma_1,$$

where, for even n, the coefficients $A_\nu^{(n)}, B_\nu^{(n)}$, and $C_\nu^{(n)}$ are equal to zero when ν is odd, and conversely.

First of all, from the first relation (17) it follows that $a_0^{(n)} = 0$. To determine the coefficients $\alpha_{\nu 1}^{(n)}, \beta_{\nu 2}^{(n)}$, and $\alpha_{\nu 2}^{(n)}$, systems of homogeneous linear equations are obtained, with determinants different from zero for $\nu > 1$. Since at the vertex D_2 of the line l_2, $\vartheta_1(0) = \vartheta_2(0) = 0$, $\alpha_{11}^{(n)}, \beta_{11}^{(n)}$, and $a_{12}^{(n)}$ will also be equal to zero.

In the cases $\beta = 1$, or $(\alpha r + r^{-1})/(r - r^{-1})$ noninteger, for determining $a_{\nu 1}^{(n)}, b_{\nu 1}^{(n)}, a_{\nu 2}^{(n)}$ $(\nu > 1)$, generally speaking, when n and ν are simultaneously even or odd, systems of nonhomogeneous linear equations with determinants different from zero are obtained; in the opposite case the systems are homogeneous.

In the case $\beta = (\alpha r + r^{-1})/(r - r^{-1})$ integer, either indeterminacy or inconsistency arises in the very system that determines these coefficients.

For determining $a_{11}^{(n)}, b_{11}^{(n)}$, and $a_{12}^{(n)}$, the following system is obtained:

a) when $\beta = 1$,

$$a_{12}^{(n)} - a_{11}^{(n)} + b_{11}^{(n)} = A_1^{(n)},$$

$$-m_2 a_{12}^{(n)} + (m_1 + 1) a_{11}^{(n)} + (m_1 - 1) b_{11}^{(n)} - k^{(n-1)} = B_1^{(n)}, \tag{18}$$

$$2 b_{11}^{(n)} r^{-1} + k^{(n-1)} r = C_1^{(n)};$$

b) when $\beta = (\alpha r + r^{-1})/(r - r^{-1})$ is noninteger,

$$a_{12}^{(n)} - a_{11}^{(n)} + b_{11}^{(n)} = A_1^{(n)},$$

$$-m_2 a_{12}^{(n)} + (m_1 + \beta) a_{11}^{(n)} + (m_1 - \beta) b_{11}^{(n)} - r(\alpha - 1) k^{(n-1)} = B_1^{(n)}, \tag{19}$$

$$(\beta - 1) r a_{11}^{(n)} - (\beta + 1) r^{-1} b_{11}^{(n)} + 2 k^{(n-1)} = C_1^{(n)}$$

with determinant equal to zero. The compatibility condition uniquely determines $k^{(n-1)}$. In addition, without essential restrictions one may take $a_{11}^{(n)} = 0$ $(n > 1)$; then $a_1^{(n)}$ and $b_{11}^{(n)}$ are also determined uniquely. This solution gives symmetric waves. Moreover, the vertical passing through the extremal point of l_2 also passes through the extremal point of l_1.

By the method of constructing majorant functions, the convergence of the various series by means of which the solution is obtained is proved. Bearing in mind that

we need majorize only trigonometric series composed of a finite number of cosines or sines; the definition of a majorant function of a special type is introduced ((4), p. 60). Solving a certain system of equations, one can obtain majorants of the functions $\vartheta_1(\sigma_1)$, $\tau_1(\sigma_1)$, $\vartheta_1(r, \sigma_1)$, $\tau_1(r, \sigma_1)$, $s(\sigma_1)$, and k in the form of holomorphic functions for sufficiently small $|\mu|$ and for any σ_1. From this, one can prove the convergence of all the series obtained by us for sufficiently small $|\mu|$.

Without detailed computations we indicate the numerical calculations up to the first three terms in case I and up to the first two terms in case II of the solution of the problem.

Case I. The wave profiles on the interface and on the free surface are given by the equations

$$-2\pi y/\lambda = \mu\cos 2\pi x/\lambda + \frac{1}{2}\mu^2\cos 4\pi x/\lambda - \mu^3\{[3(a-1)r^3/4D_3 - 5/8]\times$$
$$\times\cos 6\pi x/\lambda - [(r-r^3)/D_1 + 9/8]\cos 2\pi x/\lambda\} + [4]; \tag{20}$$

$$-2\pi y/\lambda = \ln r + \mu r\cos 2\pi x/\lambda + \mu^2\left[\frac{3}{2}r^2\cos 4\pi x/\lambda - \frac{1}{2}(r^2+1)\right] -$$
$$-\mu^3\left[((3a-1)r^3/4D_3 - 33/8)\, r^3\cos 6\pi x/\lambda - (1/2D_3)\cos 6\pi x/\lambda + \right. \tag{21}$$
$$\left. + ((r^2-1)/D_1 + 19/8)\cos 2\pi x/\lambda\right] + [4],$$

where $D_n = r^n(na-1) + (n+1)r^{-n}$ $(n = 1, 2, ...)$. In this case the crest of the wave on the interface corresponds to the crest of the wave on the free surface. The ratio between the heights b, a of the upper and lower waves is given by the expression

$$b/a = r + \frac{1}{2}\mu r(r-1) + [2]. \tag{22}$$

The wavelength is determined in terms of the velocity in the form

$$\lambda = (2\pi c^2/g)\{1 - ([2r^{-1} + (\alpha-1)r^3]/D_1)\mu^2 + [4]\}. \tag{23}$$

Case II. The wave profiles on the interface and on the free surface are given by the equations

$$-2\pi y/\lambda = \mu r(\alpha-1)\cos 2\pi x/\lambda -$$

$$-\frac{1}{2}\mu^2\left[a_{21}^{(2)}-b_{21}^{(2)}-\frac{1}{2}r^2(2\beta-\alpha-1)(\alpha-1)\right]\cos 4\pi x/\lambda+\ldots, \qquad (24)$$

$$-2\pi y/\lambda=\ln r-2\mu\cos 2\pi x/\lambda-\frac{1}{2}\mu^2\left[(r^2a_{21}^{(2)}-r^{-2}b_{21}^{(2)}+2\beta)\cos 4\pi x/\lambda-\right.$$

$$\left.-4\beta-r^2(\alpha-1)(2\beta-\alpha-1)\right]+[3],$$

where

$$a_{21}^{(2)}=\left[A'(\beta+2)r^{-2}-2m_2(\beta+2)(\beta-1)(\alpha-1)+6\beta^2(2\alpha-\beta)\right]/D_2^*,$$

$$b_{21}^{(2)}=\left[A'(\beta-2)r^{2}-2m_2r^4(\beta-1)(\alpha-1)(\beta-2)-6\beta^2(\beta-2)\right]/D_2^*,$$

$$A=-\frac{1}{2}r^2(\alpha-1)[(2\beta-\alpha-1)(3\beta-\alpha-1)-(\alpha+1)(2\beta+\alpha-3)],$$

$$D_n^*=(\beta-n)\{n(\alpha r^n+r^{-n})-\beta(r^n-r^{-n})\}\quad(n>1);$$

here the crest of the wave on the interface corresponds to the trough of the wave on the free surface. The ratio between the height of the lower wave and the depression in the upper wave is given by the expression

$$a/-b=r(\alpha-1)/2+[1]. \qquad (25)$$

From (22) and (25) we conclude that waves of the first category attain greater development on the open surface than on the interface. Conversely, internal waves of the second category are incomparably larger than external waves of the same category.

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Note: Figure translations are in progress. See original paper for figures.

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