ON THE COMPUTATION OF THE INDEX OF A SYSTEM OF ONE-DIMENSIONAL SINGULAR EQUATIONS

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Abstract

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MATHEMATICS

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ON THE COMPUTATION OF THE INDEX OF A SYSTEM OF ONE-DIMENSIONAL SINGULAR EQUATIONS

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The purpose of the present note is to show that the known formula for the index of a system of one-dimensional singular integral equations, obtained by N. I. Muskhelishvili and N. P. Vekua ([1], see also [2]), can be easily derived from simple topological considerations.

Let there be given, written in matrix form, a system of n one-dimensional singular integral equations with n unknowns:

$$a(t)u(t) + b(t)(Su)(t) + (Tu)(t) = f(t), \quad t \in \Gamma. \tag{1}$$

Here

$$(Su)(t) = \frac{1}{\pi i} \int_\Gamma \frac{u(\xi)}{\xi - t}\, d\xi;$$

Γ is a closed Lyapunov contour in the plane of the complex variable; $u(t)$ and $f(t)$ are n-component vectors; $a(t)$ and $b(t)$ are matrices of order n, continuous on Γ. We shall consider equation (1) in some Banach space B, in which: 1) the operator S is bounded and defined on the whole space; 2) if c is the operator of multiplication by a matrix $c(t)$, continuous on Γ, then the operator $cS - Sc$ is completely continuous in B. By T is denoted an operator completely continuous in B.

The index of equation (1) does not depend on the term Tu, and therefore below one may assume $T = 0$. Further, by Theorem 1 of the paper ([3]), one may restrict oneself to the case in which the contour Γ is connected.

The symbolic matrix $\Phi(t, \theta)$ of the system (1) is equal to

$$\Phi(t, \theta) = a(t) + b(t)\theta,$$

where θ is an independent variable taking only the two values 1 and -1. As usual, we assume that this matrix is nowhere degenerate on Γ. For brevity, below we shall call the symbolic matrix simply the symbol; speaking of a matrix, we shall mean by this a continuous nondegenerate matrix of order n given on Γ.

The nondegenerate continuous symbolic matrices of the given order n form a topological group with respect to multiplication. The index of the system (1) (we shall denote it by $\operatorname{Ind}\Phi$) realizes a certain homomorphism of this group into the group of integers.

Specifying the symbol $\Phi(t,\theta)$ is equivalent to specifying two matrices

$$\sigma(t) = \Phi(t,1) = a(t) + b(t); \quad \delta(t) = \Phi(t,-1) = a(t) - b(t);$$

the condition of nondegeneracy of the symbol means that the matrices $\sigma(t)$ and $\delta(t)$ are nowhere degenerate on Γ.

Since the matrices $\sigma(t)$ and $\delta(t)$ are particular values of the symbol $\Phi(t,\theta)$, when symbols are multiplied the corresponding matrices $\sigma(t)$ and $\delta(t)$ are also multiplied. We shall regard the symbol $\Phi(t,\theta)$ as an ordered

pair of matrices $\sigma(t)$ and $\delta(t)$, and write this as

$$\Phi(t,\theta) = \{\sigma(t), \delta(t)\}.$$

Then

$$\{\sigma_1(t), \delta_1(t)\}\{\sigma_2(t), \delta_2(t)\} = \{\sigma_1(t)\sigma_2(t), \delta_1(t)\delta_2(t)\}.$$

In particular, any symbol can be decomposed into a product

$$\{\sigma, \delta\} = \{\sigma, I\}\{I, \delta\}, \tag{2}$$

where I is the identity matrix.

The symbols $\{\sigma, I\}$ (and, analogously, the symbols $\{I, \delta\}$) can be identified with the corresponding matrices σ (respectively δ) in the sense that symbols of the form $\{\sigma, I\}$ (respectively $\{I, \delta\}$) form, with respect to multiplication, a group (it is a subgroup of the group of symbols) homeomorphic to the group of matrices. Therefore, if l is some homomorphism of the group of symbols into the group of integers, then

$$l(\Phi) = l(\{\sigma, I\}) + l(\{I, \delta\}) = l'\sigma + l''\delta, \tag{3}$$

where l' and l'' are some homomorphisms of the group of matrices into the group of integers.

It is easy to indicate one homomorphism of this kind: if $\gamma(t)$ is a matrix, then the named homomorphism is given by the formula

$$l_1(\gamma) = \frac{1}{2\pi} \int_\Gamma d \arg \operatorname{Det} \gamma(t). \tag{4}$$

It is also not difficult to indicate a matrix $\gamma_1(t)$ such that $l_1(\gamma_1) = 1$: it suffices to take for $\gamma_1(t)$ the diagonal matrix with elements $t, 1, \dots, 1$ along the diagonal, $\gamma_1(t) = (t, 1, \dots, 1)$. We assume here that the origin of coordinates lies inside Γ.

We shall now prove that any homomorphism l of the group of matrices into the group of integers differs from l_1 only by a constant integer factor.

Let $\gamma(t)$ be an arbitrary matrix. Decompose it into the product $\gamma(t) = C(t)U(t)$, where $C(t)$ is a self-adjoint positive definite matrix and $U(t)$ is a unitary matrix. The matrix $C(t)$ is homotopic to the identity; therefore $l(C) = l_1(C) = 0$. As is known (see, for example, ([4])), the first homotopy group of the group of unitary matrices is free cyclic. This means that the group of unitary matrices defined and continuous on the contour Γ, which is homeomorphic to a circle, can be decomposed into disjoint homotopy classes having the form ω^m, $m = \dots, -2, -1, 0, 1, 2, \dots$, where ω is some homotopy class not containing the identity. If the matrix $U(t)$ belongs to the homotopy class ω^m, then $l(\gamma) = ml(\omega)$ and $l_1(\gamma) = ml_1(\omega)$; moreover, from the equality $l_1(\gamma_1) = 1$ it follows that $l_1(\omega) = \pm 1$. Hence $l(\gamma) = \mu l_1(\gamma)$, where $\mu = \pm l(\omega)$, and our assertion is proved.

Thus, in formula (3),

$$l'(\sigma) = \mu' l_1(\sigma), \qquad l''(\delta) = \mu'' l_1(\delta),$$

where μ' and μ'' are integer constants. Applying formula (3) to the case $l(\Phi) = \operatorname{Ind} \Phi$, we have

$$\operatorname{Ind} \Phi = \mu' l_1(\sigma) + \mu'' l_1(\delta) = \frac{\mu'}{2\pi} \int_\Gamma d \arg \operatorname{Det} \sigma(t) + \frac{\mu''}{2\pi} \int_\Gamma d \arg \operatorname{Det} \delta(t). \tag{5}$$

In system (1) put $b(t) = 0$, $a(t) = \gamma_1(t)$, where $\gamma_1(t)$ is the matrix introduced above. System (1) then reduces to a linear algebraic system whose index is, obviously, equal to zero. At the same time $\sigma(t) = \delta(t) = \gamma_1(t)$ and

$$\int_\Gamma d \arg \operatorname{Det} \sigma(t) = \int_\Gamma d \arg \operatorname{Det} \delta(t) = 2\pi.$$

It now follows from formula (5) that $\mu'' = -\mu'$. Denoting, for brevity, $\mu' = \mu$, we have

$$\operatorname{Ind}\Phi = \frac{\mu}{2\pi}\int_\Gamma d\arg\operatorname{Det}\left[\sigma^{-1}(t)\delta(t)\right]. \tag{6}$$

Now put in system (1)

$$a(t) = \left(\frac{t+\bar{t}}{2},\, 1, \dots, 1\right),$$

$$b(t) = \left(\frac{t-\bar{t}}{2},\, 0, \dots, 0\right),$$

so that

$$\sigma(t) = \gamma_1(t), \qquad \delta(t) = \gamma_1^*(t).$$

The index of system (1) coincides in this case with the index of a single singular equation

$$\frac{t+\bar{t}}{2}\,\varphi(t) + \frac{t-\bar{t}}{2}\,(S\varphi)(t) = 0,$$

and this latter index is easily computed and is equal to −2. Formula (6) then gives $\mu = 1$, and we arrive at the formula of N. I. Muskhelishvili—N. P. Vekua:

$$\operatorname{Ind}\Phi = \frac{1}{2\pi}\int_\Gamma d\arg\operatorname{Det}\left[\sigma^{-1}(t)\delta(t)\right].$$

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