ON NONTRIVIAL CONSTRUCTIVE MAPPINGS OF CERTAIN SETS

MATHEMATICS

1966

SovietRxiv

Abstract

Full Text

UDC 51.01:518.5

MATHEMATICS

V. A. SHURYGIN

ON NONTRIVIAL CONSTRUCTIVE MAPPINGS OF CERTAIN SETS

(Presented by Academician P. S. Novikov, 11 VIII 1965)

The present note belongs to the constructive trend in mathematics (^{1}). The term **set** is used here in the same sense as in papers (^{2},^{3}). We shall assume that, for the elements of each of the sets considered here, a reflexive, symmetric, and transitive equality relation has been introduced (by means of a two-place predicate), and elements that stand in the equality relation will be called **equal**. If $\mathfrak{M}$ is a set in an alphabet A, then A will be called the **alphabet of the set** $\mathfrak{M}$. Let $\mathfrak{M}$ and $\mathfrak{N}$ be two sets. By a **partial mapping** of the set $\mathfrak{M}$ into the set $\mathfrak{N}$ we shall mean any normal algorithm (^{4}) over the union of the alphabets of these sets that transforms those elements of the set $\mathfrak{M}$ to which it is applicable into elements of the set $\mathfrak{N}$. A partial mapping $\mathfrak{A}$ of the set $\mathfrak{M}$ into the set $\mathfrak{N}$ will be called a **complete mapping** of $\mathfrak{M}$ into $\mathfrak{N}$ if the algorithm $\mathfrak{A}$ is applicable to every element of the set $\mathfrak{M}$. A partial mapping $\mathfrak{A}$ of the set $\mathfrak{M}$ into the set $\mathfrak{N}$ will be called an **equality-preserving mapping** if, for any elements P and Q of the set $\mathfrak{M}$, from the facts that P and Q are equal elements and that the algorithm $\mathfrak{A}$ is applicable both to P and to Q, it follows that the elements $\mathfrak{A}(P)$ and $\mathfrak{A}(Q)$ of the set $\mathfrak{N}$ are also equal. A partial mapping $\mathfrak{A}$ of the set $\mathfrak{M}$ into the set $\mathfrak{N}$ will be called **nontrivial** if it is false that there do not exist elements P and Q of the set $\mathfrak{M}$ such that the algorithm $\mathfrak{A}$ is applicable both to P and to Q and the elements $\mathfrak{A}(P)$ and $\mathfrak{A}(Q)$ of the set $\mathfrak{N}$ are not equal. A **trivial** partial mapping will be one that is not nontrivial.

There are known examples of such sets $\mathfrak{M}, \mathfrak{N}$ that every complete mapping of $\mathfrak{M}$ into $\mathfrak{N}$ preserving equality transforms all elements of the set $\mathfrak{M}$ into elements of the set $\mathfrak{N}$ equal to one another, i.e. every complete mapping of $\mathfrak{M}$ into $\mathfrak{N}$ preserving equality is trivial. Thus, for example, every complete mapping of the set of constructive real numbers—duplexes (^{3})—into the set of integers that transforms equal real numbers into equal integers is trivial. This follows from the known properties of constructive functions of a real variable (see Corollary {3} in § 4 in (^{5})). Here we shall give other analogous examples, and moreover shall establish the impossibility not only of complete nontrivial equality-preserving mappings, but also of nontrivial partial mappings preserving equality and satisfying certain conditions. These examples are based on the result formulated in Theorem 1.

Let $\mathfrak{M}$ be a set and P an element of this set. A normal algorithm $\mathfrak{A}$ over the alphabet of this set will be called an **algorithm complete on P in $\mathfrak{M}$** if it is applicable to every element of the set $\mathfrak{M}$ equal to the element P. In the same sense we shall use the term **partial mapping complete on P in $\mathfrak{M}$**. A normal algorithm $\mathfrak{A}$ over the alphabet of the set $\mathfrak{M}$ will be called an **algorithm empty on P in $\mathfrak{M}$** if it is not applicable to any element of the set $\mathfrak{M}$ equal to the element P. An element P of the set $\mathfrak{M}$ will be called a **node** if, for no element Q of this set, there exists a normal algorithm complete on P in $\mathfrak{M}$ and empty on Q

in $\mathfrak{M}$. We shall say that a set $\mathfrak{M}$ has an **enumerable equality condition** if there exists a normal algorithm applicable to a pair of elements P, Q of the set $\mathfrak{M}$ if and only if the elements P and Q are equal. We shall say that a set $\mathfrak{M}$ has an **enumerable inequality condition** if there exists a normal algorithm applicable to a pair of elements P, Q of the set $\mathfrak{M}$ if and only if the elements P and Q are not equal.

Theorem 1. *Let $\mathfrak{M}$ be a set and P a nodal element of this set. Whatever the set $\mathfrak{N}$, having an enumerable equality condition or an enumerable inequality condition, it is impossible to construct a nontrivial partial mapping of $\mathfrak{M}$ into $\mathfrak{N}$ that preserves equality and is complete on P in $\mathfrak{M}$.*

We shall use the term **quasinumber** in the same sense as in [5]. Let $\mathfrak{M}$ be a set. We shall say that $\mathfrak{M}$ is an **invariant set of quasinumbers** if the following conditions hold: 1) every element of the set $\mathfrak{M}$ is a quasinumber; 2) for any quasinumbers x and y, from the fact that x and y are conditionally equal and x belongs to $\mathfrak{M}$, it follows that y also belongs to $\mathfrak{M}$. The elements of an invariant set of quasinumbers that are conditionally equal quasinumbers will be called **equal elements**.

Theorem 2. *Whatever the invariant set of quasinumbers, every quasinumber belonging to this set is a nodal element of this set.*

The set of constructive real numbers (whose equal elements are taken to be equal constructive real numbers) has an enumerable inequality condition; therefore, from Theorems 1 and 2 we obtain

Corollary 1. *Whatever the invariant set $\mathfrak{M}$ of quasinumbers, it is impossible to construct a nontrivial partial mapping of the set $\mathfrak{M}$ into the set of constructive real numbers that preserves equality and is applicable to every quasinumber conditionally equal to some quasinumber from $\mathfrak{M}$.*

By a **nondecreasing quasinumber** we shall mean any quasinumber whose basis is a nondecreasing sequence of rational numbers. Let $\mathfrak{M}$ be a set. We shall say that $\mathfrak{M}$ is an **invariant set of nondecreasing quasinumbers** if the following conditions hold: 1) every element of the set $\mathfrak{M}$ is a nondecreasing quasinumber; 2) for any nondecreasing quasinumbers x and y, from the fact that x and y are conditionally equal and x belongs to $\mathfrak{M}$, it follows that y also belongs to $\mathfrak{M}$.

Let $\mathfrak{M}$ be an invariant set of nondecreasing quasinumbers and let x be an element of this set. We shall say that x is a **minimal element** of the set $\mathfrak{M}$ if $\mathfrak{M}$ contains no quasinumber conditionally less than x. We shall say that x is a **maximal element** of $\mathfrak{M}$ if $\mathfrak{M}$ contains no quasinumber conditionally greater than x.

Theorem 3. *The minimal element of any invariant set of nondecreasing quasinumbers is a nodal element of this set.*

Corollary 2. *Whatever the invariant set $\mathfrak{M}$ of nondecreasing quasinumbers, it is impossible to construct a nontrivial partial mapping of this set into the set of constructive real numbers that preserves equality and is applicable to all nondecreasing quasinumbers conditionally equal to some nondecreasing quasinumber from $\mathfrak{M}$ that is not a maximal element of the set $\mathfrak{M}$.*

In particular, there is no algorithm that constructs constructive limits for each of those increasing bounded-above sequences of rational numbers for which constructive limits cannot fail to exist.

Let A be an alphabet, and let α and β be distinct letters not belonging to A. Let $\mathfrak{M}$ be a set. We shall say that $\mathfrak{M}$ is a **set**

normal algorithms over the alphabet A, if every element of the set $\mathfrak{M}$ is a record of some normal algorithm in the alphabet $A \cup \{\alpha, \beta\}$.

Let $\mathfrak{M}$ be a set of normal algorithms over the alphabet A. We shall say that the elements P and Q of the set $\mathfrak{M}$ are equal if the normal algorithms $\langle P \rangle$ and $\langle Q \rangle$ are equivalent relative to A (here $\langle R \rangle$ denotes the normal algorithm in the alphabet $A \cup \{\alpha, \beta\}$ whose record is the word R). We shall say that $\mathfrak{M}$ is invariant if the following condition is satisfied: whatever the element P of the set $\mathfrak{M}$ and the normal algorithm $\mathfrak{A}$ in the alphabet $A \cup \{\alpha, \beta\}$, from the fact that $\langle P \rangle$ and $\mathfrak{A}$ are equivalent relative to A it follows that the record of the algorithm $\mathfrak{A}$ is an element of $\mathfrak{M}$.

Let $\mathfrak{A}$ and $\mathfrak{B}$ be normal algorithms over the alphabet A. We shall say that $\mathfrak{A}$ is an extension of $\mathfrak{B}$ relative to A, if for every word P in A, from the fact that $\mathfrak{B}$ is applicable to P, it follows that $\mathfrak{A}$ is applicable to P and that $\mathfrak{A}(P) = \mathfrak{B}(P)$.

Let $\mathfrak{M}$ be a set of normal algorithms over the alphabet A. We shall say that an element P of the set $\mathfrak{M}$ is minimal if for every element Q of the set $\mathfrak{M}$ the algorithm $\langle Q \rangle$ is an extension relative to A of the algorithm $\langle P \rangle$.

Theorem 4. *The minimal element of any invariant set of normal algorithms over the alphabet A is a nodal element of this set.*

Corollary 3. *Let $\mathfrak{M}$ be an invariant set of normal algorithms over the alphabet A, and let P be its minimal element. It is impossible to construct a nontrivial partial mapping of the set $\mathfrak{M}$ into the set of constructive real numbers that preserves equality and is total on P in $\mathfrak{M}$.*

Let B be an alphabet and $\mathfrak{N}$ a set. We shall say that $\mathfrak{N}$ is a set of associative calculi in the alphabet B ((^{4}), Chap. 6, § 1), if every element of the set $\mathfrak{N}$ is a record of a defining system of relations of some associative calculus in B ((^{4}), p. 345; henceforth, instead of the term "record of a defining system of relations of an associative calculus," we shall use the term "record of an associative calculus").

Let $\mathfrak{N}$ be a set of associative calculi in the alphabet B. We shall say that the elements P and Q of the set $\mathfrak{N}$ are equal if the associative calculi in B whose records are the words P and Q are isomorphic. From the results of [6] it follows that any set of associative calculi in B has an enumerable equality condition. Therefore, from Theorems 1 and 4 we obtain

Corollary 4. *Let A and B be alphabets, $\mathfrak{M}$ an invariant set of normal algorithms over the alphabet A, P the minimal element of the set $\mathfrak{M}$, $\mathfrak{N}$ a set of associative calculi in B, and $\mathfrak{A}$ a partial mapping of $\mathfrak{M}$ into $\mathfrak{N}$ that preserves equality and is total on P in $\mathfrak{M}$. If the elements Q and R of the set $\mathfrak{M}$ are such that $\mathfrak{A}(Q)$ and $\mathfrak{A}(R)$ are defined, then $\mathfrak{A}(Q)$ and $\mathfrak{A}(R)$ are equal elements of the set $\mathfrak{N}$.*

In Corollaries 3 and 4, as $\mathfrak{M}$ one may take the set of all normal algorithms over A. The minimal element of this set is the record of a normal algorithm that is not applicable to any word in A.

The author expresses gratitude to A. A. Markov for the problem he proposed, work on which led to the results described here, and to N. M. Nagorny for valuable advice and critical remarks.

Received
6 VII 1965

References

1. A. A. Markov, Tr. Matem. inst. im. V. A. Steklova AN SSSR, **67** (1962).
2. N. A. Shanin, ibid., **52** (1958).
3. N. A. Shanin, ibid., **67** (1962).
4. A. A. Markov, ibid., **42** (1954).
5. G. S. Tseitin, ibid., **67**, 362 (1962).
6. N. M. Nagorny, *Zs. math. Logik u. Grundlagen d. Math.*, **6** (1960).

Note: Figure translations are in progress. See original paper for figures.

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.