Soviet-era science, translated into English

SOME EXTREMAL PROBLEMS OF CONFORMAL MAPPING OF DOUBLY CONNECTED AND MULTIPLY CONNECTED DOMAINS

MATHEMATICS

1966

SovietRxiv

View the original and related papers at https://sovietrxiv.org/items/ru-196601.96386

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.

Abstract

Full Text

UDC 517.54

MATHEMATICS

P. M. TAMRAZOV

SOME EXTREMAL PROBLEMS OF CONFORMAL MAPPING OF DOUBLY CONNECTED AND MULTIPLY CONNECTED DOMAINS

(Presented by Academician V. I. Smirnov on 2 I 1966)

§1. Let a doubly connected domain D be fixed, containing the point $z = 0$ and bounded by the unit circle C_1 and a radial slit Γ with endpoints at the points $z = q_0$ and $z = p_0$ $(0 < q_0 < p_0 < 1)$. By J denote the class of schlicht conformal mappings of the domain D onto the unit disk, taking the point $z = 0$ to $w = 0$ and the circle C_1 onto the outer boundary of the image. Let $c > 0$, and let J_c denote the subclass of the class J consisting of mappings $f(z)$ satisfying the condition $|f'(0)| = c$. By $f_0(z)$ denote the mapping in the class J that maps C_1 onto itself and Γ onto a circular arc Γ_0 with center at the point $w = 0$ and midpoint on the positive real axis. The angular measure of the arc Γ_0 will be denoted by $2\theta_0$. Also denote $|f_0'(0)| = c_0$. For $f \in J$, by $p(f)$ and $q(f)$ denote, respectively, the maximum and minimum of the quantity $|w|$ for $w \in f(\Gamma)$.

§2. We state the results of the investigation of the problems of $\max p(f)$ and $\min q(f)$ in the classes J_c for various $c > 0$.

It is known (^{1,2}) that for $c > c_0$ the classes J_c are empty and that the class J_{c_0} consists of mappings of the form $f(z) \equiv e^{i\theta} f_0(z)$ (θ real).

The solution of the problems formulated above for the class J_1 is a consequence of our previous results (^{3,4}).

Theorem 1. *The problems of* $\max p(f)$ *and* $\min q(f)$ *in the class* J_1 *are solved by the functions* $f(z) \equiv e^{i\theta} z$ *(θ a real number), and only by them.*

The case $c \in (0, 1)$ is formulated in terms of the function $\mu_c(z)$, which satisfies the conditions $\mu_c(0) = 0$, $\mu_c'(0) = c$, and maps the unit disk K schlichtly and conformally onto the domain obtained from K by making a radial slit adjacent to the point $w = 1$.

Theorem 2. *For* $c \in (0, 1)$*, in the class* J_c *the problem of* $\max p(f)$ *is solved by the functions* $f(z) \equiv e^{i\theta} \mu_c(-z)$*, while the problem of* $\min q(f)$ *is solved by the* *functions* $f(z) \equiv e^{i\theta}\mu_c(z)$, *and only by them.*

In the case $c = (1, c_0)$, the extremal mappings have another qualitative character, and the solution is connected with an essential use of the theory of the quadratic differential and the general Jenkins theorem on coefficients (^{5}). First one must describe the classes of extremals; moreover, we shall do this only for the problem of $\max p(f)$ (see the lemma and Theorems 3-5).

Let

$$J_{(1,c_0)} = \bigcup_{c=(1,c_0)} J_c.$$

If the numbers a and b are such that $0 < |a| < |b| < 1$, then by $J^{a,b}$ denote the class of mappings $f \in J_{(1,c_0)}$ for each of which the domain $f(D)$ does not contain the point $w = b$ and is admissible in the unit disk K with respect to the positive (in K) quadratic differential (for terminology see (^{5}))

$$Q^{a,b}(w)\,dw^2 = -\frac{(w-a)(w-1/\bar{a})}{w^2(w-b)(w-1/\bar{b})}\,dw^2.$$

We also denote

$$J^0 = \bigcup_{\substack{a,b \\ (0<|a|<|b|<1)}} J^{a,b}; \qquad J_c^{a,b} = J_c \cap J^{a,b}; \qquad J_c^0 = J_c \cap J^0.$$

Theorem 3. On the interval $(1, c_0)$ there exist two positive functions (of c) $a(c)$ and $b(c)$ such that $a(c) < b(c) < 1$, and the class $J_c^{a(c),\,b(c)}$ is nonempty for every $c \in (1, c_0)$. The functions $a(c)$ and $b(c)$ are uniquely determined by the domain D, and the classes $J_c^{a(c),\,b(c)}$ are unique in the following sense: if, for some values a, b, and c, the class $J_c^{a,b}$ is nonempty, then it necessarily follows that $\arg a = \arg b$, $|a| = a(c)$, $|b| = b(c)$, and, moreover, if $f \in J_c^{a,b}$, then $e^{-i\arg a} f(z) \in J_c^{a(c),\,b(c)}$. The functions $a(c)$ and $b(c)$ are continuous on the interval $(1, c_0)$, and the second of them is strictly decreasing.

Lemma. Let $0 < a < b < 1$. Then the quadratic differential $Q^{a,b}(w)dw^2$ is positive in the unit disk K and has no critical points there other than the points $w = 0$, $w = a$, and $w = b$. One of the trajectories (denote it by T) is the interval (a, b); one of the trajectories (denote it by T_1) closes at the point $w = a$. The set $K \setminus (T \cup T_1)$ consists of one circular domain (containing the point $w = 0$) and one annular domain (whose outer boundary is the unit circle). Each of the trajectories in the circular and annular domains, and the closure of the trajectory T_1, are Jordan curves lying entirely in K, enclosing the point $w = 0$, and starlike with respect to it; moreover, if $\rho = \rho(\varphi)$ is the polar equation of any such trajectory (in the natural polar coordinate system in the w-plane), then the function $\rho(\varphi)$ is even and strictly decreases on the interval $(0, \pi)$. Therefore, for the trajectory T_1, for all $\varphi \in (0, 2\pi)$ the inequalities $\rho(\varphi) < a < b$ hold.

Note that, according to Theorem 3, it suffices to study only the class

$$J^+ = \bigcup_{c \in (1, c_0)} J_c^{a(c), b(c)}.$$

Define, on the class of functions regular in a neighborhood of the point $z = 0$, the functional $\Psi(f) = f'(0)$.

Below, in classes of analytic functions, only the natural topology determined by uniform convergence inside the domain is considered.

Theorem 4. On the interval $(1, c_0)$ there exists a (unique) function (of c) $\theta(c)$, $0 < \theta(c) < \pi$, such that the restriction of $\Psi(f)$ to the class J^+ is a homeomorphism of this class onto the point set

$$M = \{t: \ 1 < |t| < c_0, \ -\theta(|t|) \leq \arg t \leq \theta(|t|)\}.$$

This homeomorphism admits a continuation to the (compact) closure of the class J^+; moreover, the complement of the class J^+ to its closure consists of the mappings $f(z) \equiv z$ and $f(z) \equiv e^{i\theta} f_0(z)$, $-\theta_0 \leq \theta \leq \theta_0$. The function $\theta(c)$ is continuous and strictly increasing on the interval $(1, c_0)$ from the value 0 to the value θ_0.

From Theorem 4 it is evident that the functions of the class J^+ can be parametrized by the points of the set M (which is the range of values of the functional Ψ on the class J^+). It is clear how to do this: if $t \in M$, then by $f(t, z)$ we denote that function $f \in J^+$ for which $\Psi(f) = t$.

Theorem 5. If $\delta \in (-2\pi, 2\pi)$, then the identity $f(t_1, z) \equiv f(t_2, z) e^{i\delta}$ holds only when $\delta = 0$, $t_1 = t_2$. Let $t = ce^{i\theta}$, $c > 0$, $-\pi < \theta < \pi$. If the point $ce^{i\theta}$ is an interior point of the set M, then the inner boundary of the image of the domain D under the mapping $w = f(ce^{i\theta}, z)$ consists of exactly three nondegenerate arcs meeting at the point $w = a(c)$ at angles $2/3\pi$; moreover, two of these arcs, γ' and γ'', lie on the trajectory T_1 (γ' adjoins the point $w = a(c)$ from above, and γ'' from below), while the third arc γ is the closure of the trajectory T. For $\theta = \theta(c)$ the arc γ' is maximal and the arc γ'' degenerates; for $\theta = -\theta(c)$ conversely. For $\theta = 0$ (and only for this value) γ' and γ'' are symmetric to one another with respect to the real axis, $\overline{f(c, \bar{z})} \equiv f(c, z)$, and the functions $a(c)$ and $b(c)$ satisfy the cor-

relations:

$$a(c) = f(c, q_0); \quad b(c) = f(c, p_0); \quad \lim_{c \to 1} a(c) = q_0; \quad \lim_{c \to 1} b(c) = p_0;$$

$$\lim_{c \to c_0} a(c) = \lim_{c \to c_0} b(c) = q(f_0) = p(f_0).$$

The number $2\theta(c)$ is equal to the sum of the lengths of the boundary arcs γ' and γ'' of the domain $f(ce^{i\theta}, D)$ in the $Q^{a(c),b(c)}$-metric.

Finally, on the inner boundary of the domain $f(ce^{i\theta}, D)$ the relation $|w| \leq b(c)$ holds, and the equality sign in it occurs only for the point $w = b(c)$.

Theorem 6. For any $c \in (1, c_0)$ and $f \in J_c$ the relation $p(f) \leq b(c)$ holds, with equality if and only if there is a pair of real numbers θ and δ, $-\theta(c) \leq \theta \leq \theta(c)$, $-\pi < \delta \leq \pi$, such that $ce^{i\theta} \in M$ and

$$f(z) \equiv f(ce^{i\theta}, z)e^{i\delta}.$$

For each function $f \in J_c$ there can exist at most one pair of real numbers θ and δ $(-\pi < \theta < \pi,\ -\pi < \delta \leq \pi)$ satisfying the indicated identity.

Analogous results are also valid for the problem on $\min q(f)$ in the classes J_c, $c \in (1, c_0)$.

§3. The results set forth above carry over to triply connected domains, which it is convenient to regard as bounded by two concentric circles and a radial slit, with each boundary circle being allowed to degenerate into a point. The basic scheme of the investigation is preserved here; however, in the case when both boundary circles are nondegenerate, we can no longer use Jenkins' general theorem on coefficients, since the corresponding quadratic differential has no poles of order higher than one. In place of the indicated theorem we have established a result which, for quadratic differentials without multiple poles, plays the role of a general theorem on coefficients.

Let us note that all the preceding results can be generalized to multiply connected (including infinitely connected) domains with one, two, or three distinguished boundary components (4), and also to quasiconformal mappings of such domains.

§4. Let the conditions of § 1 again be satisfied. Denote by J^* the subclass of the class J consisting of all mappings that carry the unit circle onto itself.

Theorem 7. The problem of $\max q(f)$ in the class J and the problem of $\min p(f)$ in the class J^* are solved by the functions

$$f(z) \equiv f_0(z)e^{i\theta}$$

(θ is a real number) and only by them.

The assertion of Theorem 7 on $\min p(f)$ is also true in a broader class of functions, consisting of all univalent conformal mappings of the domain D that carry the point $z = 0$ to $w = 0$ and the unit circle to the set $|w| \geq 1$.

Institute of Mathematics
Academy of Sciences of the Ukrainian SSR

Received
28 XII 1965

REFERENCES

[1] H. Grötzsch, Leipz. Ber., **81**, 38, 247 (1929).
[2] E. Rengel, Jahresber. Deutsch. math. Ver., **44**, 51 (1934).
[3] P. M. Tamrazov, Dokl. Ukr. SSR, No. 5, 563 (1962).
[4] P. M. Tamrazov, Matem. sborn., **67** (109), 329 (1965).
[5] J. Jenkins, *Univalent Functions and Conformal Mappings*, IL, 1962.

Note: Figure translations are in progress. See original paper for figures.

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.