ON SLIP SURFACES

THEORY OF ELASTICITY

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Abstract

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THEORY OF ELASTICITY

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ON SLIP SURFACES

IN THREE-DIMENSIONAL RIGID-PLASTIC BODIES

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The properties of the equations of the theory of ideal plasticity for a stressed state corresponding to an edge of Tresca's prism were considered in papers ([1], [2]). Surfaces of discontinuity of strain rates under the Mises plasticity condition were investigated by Thomas ([3]). Characteristic manifolds of the equations of the theory of ideal plasticity under an arbitrary piecewise-linear plasticity condition were obtained in ([4]). Below we consider the kinematic relations that must be satisfied on slip surfaces, which are characteristic under the Tresca plasticity condition; equations are obtained which must be satisfied by discontinuities of velocities on slip surfaces.

By a slip surface in what follows is meant a surface on which the material experiences maximum pure shear. We shall show that surfaces of discontinuity of velocities and surfaces of discontinuity of strain rates in an incompressible material coincide with slip surfaces.

On the surface of discontinuity of displacement velocities, for the strain rates we have ([5])

$$\varepsilon_{ij} = \Psi\left([u_i]\nu_j + [u_j]\nu_i\right), \tag{1}$$

where $[u_i]$ are jumps of velocities, ν_i is the unit vector of the normal to the surface of discontinuity, and ψ is a certain proportionality factor. Multiplying (1) by ν_j and taking into account that in an incompressible material $[u_i]\nu_i = 0$, we obtain

$$[u_i]\psi = \varepsilon_{ij}\nu_j. \tag{2}$$

Eliminating $[u_i]$ from (1) with the aid of formula (2), we obtain that on the surface of discontinuity of the strain rate the relations

$$\varepsilon_{ij} = \varepsilon_{ik}\nu_j\nu_k + \varepsilon_{jk}\nu_k\nu_i \tag{3}$$

must be satisfied. Choosing a canonical coordinate system ($\nu_1 = \nu_2 = 0,\ \nu_3 = 1$), we obtain that on the surface of discontinuity of velocities

$$\varepsilon_{11} = \varepsilon_{22} = \varepsilon_{33} = \varepsilon_{12} = 0, \qquad \varepsilon_{13} \neq 0,\ \varepsilon_{23} \neq 0,$$

i.e., the surface of discontinuity of velocities coincides with the slip surface.

On the surface of discontinuity of deformations (the velocities are continuous), from the geometrical compatibility conditions we have

$$2[\varepsilon_{ij}] = \lambda_i\nu_j + \lambda_j\nu_i. \tag{4}$$

From the formal coincidence of formulas (1) and (4) we conclude that on the surface of discontinuity of strain rates

$$[\varepsilon_{ij}] = [\varepsilon_{ik}]\nu_k\nu_j + [\varepsilon_{jk}]\nu_k\nu_i. \tag{5}$$

For a smooth yield surface

$$[\varepsilon_{ij}] = [\lambda]\,\partial f/\partial\sigma_{ij} = \varphi\varepsilon_{ij}, \tag{6}$$

since on the surface of discontinuity of strain rates the stresses are continuous. From (5) and (6) we conclude that the surfaces of discontinuity of velocities

deformations for smooth yield surfaces coincide with slip surfaces. Let us note that, in the presence of an angular point on the yield surface, this coincidence is not necessary. From relations (3), if in the latter the coordinate axes are taken to coincide with the principal directions of the tensor ε_{ij}, it follows that

$$\varepsilon_i = \varepsilon_j, \qquad \varepsilon_k = 0, \qquad \nu_i = \pm\nu_j = \pm\sqrt{2}/2, \qquad \nu_k = 0. \tag{7}$$

It follows from relations (7) that, for any stress state satisfying the plasticity condition, slip surfaces can be realized only under the Tresca condition; in this case the slip surfaces will be characteristic surfaces.

Let us write down the relations that must be satisfied by the velocities on a slip surface.

The strain rates are expressed in terms of the displacements by the Cauchy formulas

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right). \tag{8}$$

For the partial derivatives of the velocities we have

$$\frac{\partial u_i}{\partial x_j} = \frac{du_i}{dn}\nu_j + g^{\alpha\beta}\frac{\partial u_i}{\partial y_\alpha}\frac{\partial x_j}{\partial y_\beta}, \tag{9}$$

where $g^{\alpha\beta}$ is the contravariant metric tensor of the slip surface, y_α $(\alpha, \beta = 1, 2)$ are curvilinear coordinates on the surface, and $x_i(y_\alpha, y_\beta)$ is the equation of the slip surface.

Substituting (9) into relations (8), we obtain

$$2\varepsilon_{ij} = \frac{du_i}{dn}\nu_j + \frac{du_j}{dn}\nu_i + g^{\alpha\beta}\left(\frac{\partial u_i}{\partial y_\alpha}\frac{\partial x_j}{\partial y_\beta} + \frac{\partial u_j}{\partial y_\alpha}\frac{\partial x_i}{\partial y_\beta}\right). \tag{10}$$

Equating in (10) the indices i and j, we obtain

$$\frac{du_i}{dn}\nu_i = -\,g^{\alpha\beta}\frac{\partial u_i}{\partial y_\alpha}\frac{\partial x_i}{\partial y_\beta}. \tag{11}$$

Multiplying (10) by ν_j, we obtain

$$\frac{du_i}{dn} = 2\varepsilon_{ij}\nu_j + g^{\alpha\beta}\left(\frac{\partial u_k}{\partial y_\alpha}\frac{\partial x_k}{\partial y_\beta}\nu_i - \frac{\partial u_j}{\partial y_\alpha}\frac{\partial x_i}{\partial y_\beta}\nu_j\right). \tag{12}$$

Substituting (12) into (10) and taking into account that equality (3) holds on the slip surface, we obtain

$$2g^{\alpha\beta}\frac{\partial u_k}{\partial y_\alpha}\frac{\partial x_k}{\partial y_\beta}\nu_i\nu_j + g^{\alpha\beta}\left(\frac{\partial u_i}{\partial y_\alpha}\frac{\partial x_j}{\partial y_\beta} + \frac{\partial u_j}{\partial y_\alpha}\frac{\partial x_i}{\partial y_\beta}\right) = g^{\alpha\beta}\frac{\partial u_k}{\partial y_\alpha}\nu_k\left(\frac{\partial x_i}{\partial y_\beta}\nu_j + \frac{\partial x_j}{\partial y_\beta}\nu_i\right). \tag{13}$$

Equating in (13) the indices i and j, we obtain

$$g^{\alpha\beta}\frac{\partial u_k}{\partial y_\alpha}\frac{\partial x_k}{\partial y_\beta} = 0,$$

and relations (13) take the form

$$g^{\alpha\beta}\left(\frac{\partial u_i}{\partial y_\alpha}\frac{\partial x_j}{\partial y_\beta} + \frac{\partial u_j}{\partial y_\alpha}\frac{\partial x_i}{\partial y_\beta}\right) = g^{\alpha\beta}\frac{\partial u_k}{\partial y_\alpha}\nu_k\left(\frac{\partial x_i}{\partial y_\beta}\nu_j + \frac{\partial x_j}{\partial y_\beta}\nu_i\right). \tag{14}$$

Multiplying equality (14) by ν_j, we are convinced that the obtained relations are satisfied identically, i.e., among the 6 equalities (14) only 3 will be independent.

We obtain the independent equalities from relations (14) by multiplying them by $\partial x_i/\partial y_\sigma$, $\partial x_j/\partial y_\tau$; we have

$$\frac{\partial u_i}{\partial y_\sigma}\frac{\partial x_i}{\partial y_\tau} + \frac{\partial u_i}{\partial y_\tau}\frac{\partial x_i}{\partial y_\sigma} = 0. \tag{15}$$

The vector u_i can always be represented in the form

$$u_i = u_n v_i + u^\alpha \partial x_i/\partial y_\alpha, \tag{16}$$

where u_n is the component of the velocity normal to the surface, and u^α are the contravariant tangential components of the velocity vector.

From (15)–(16) it follows that

$$\frac{\partial u^\alpha}{\partial y_\tau}g_{\sigma\alpha} + \frac{\partial u^\alpha}{\partial y_\sigma}g_{\tau\alpha} - 2b_{\sigma\tau}u_n = 0. \tag{17}$$

In equation (17) the unknown quantity u_n enters algebraically; multiplying (17) by $g^{\sigma\tau}$, we obtain

$$2\Omega u_n = \partial u^\alpha/\partial y_\alpha, \tag{18}$$

where Ω is the mean curvature of the slip surface.

Eliminating the quantity u_n from (17), we obtain

$$\Omega\frac{\partial u^\alpha}{\partial y_\tau}g_{\sigma\alpha} + \Omega\frac{\partial u^\alpha}{\partial y_\sigma}g_{\tau\alpha} - b_{\sigma\tau}\frac{\partial u^\alpha}{\partial y_\alpha} = 0. \tag{19}$$

Among the 3 equations (19), only 2 are independent, as is easily verified by multiplying the relations (19) by $g^{\sigma\tau}$. Thus, to determine the velocities on the slip surface it is necessary to solve two equations with two unknown functions u^α, depending on two variables.

The characteristic directions of the system of equations (19) are the asymptotic directions of the surface, and the system of equations (19) will be hyperbolic if the Gaussian curvature of the surface is $K = \varkappa_1\varkappa_2 < 0$; elliptic if $K > 0$, and parabolic if $K = 0$. If $K < 0$, then the characteristic lines may be chosen as curvilinear coordinates on the surface, and, taking into account that in this case $b_{11} = b_{22} = 0$ ((6), § 6, Chap. XVI), the relations (19) take the form

$$\frac{\partial u^\alpha}{\partial y_\tau} g_{\tau\alpha} = 0 \quad \text{(do not sum over } \tau\text{)}. \tag{20}$$

If at a given point of the surface $g_{11} = g_{22} = 1$, then the components of the contravariant vector u^α coincide with the covariant components, and the relations (20) take the form

$$\frac{\partial u_1}{\partial s_1} + g_{12}\frac{\partial u_2}{\partial s_1} = 0, \qquad \frac{\partial u_2}{\partial s_2} + g_{12}\frac{\partial u_1}{\partial s_2} = 0, \qquad g_{12} = \sqrt{\frac{2\Omega^2}{K + 2\Omega^2}}. \tag{21}$$

On surfaces of velocity discontinuity, the relations (17) must hold on the right and on the left of the discontinuity surface, and, taking into account that $[u_n] = 0$, we obtain

$$\frac{\partial [u^\alpha]}{\partial y_\tau} g_{\sigma\alpha} + \frac{\partial [u^\alpha]}{\partial y_\sigma} g_{\tau\alpha} = 0. \tag{22}$$

The relations (22) define a system of three equations which the velocity jumps on the slip surface must necessarily satisfy.

Let us note that, in the case of plane deformation, the relations (17) become the Geiringer relations, while the relations (22) assert that $[u] = \text{const}$ on the slip surface.

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