ON SUBSPACES AND BASES IN SPACES OF CONTINUOUS FUNCTIONS

MATHEMATICS

1966

SovietRxiv

Abstract

Full Text

UDC 513.88:513.83

MATHEMATICS

V. I. GURARII

ON SUBSPACES AND BASES IN SPACES OF CONTINUOUS FUNCTIONS

(Presented by Academician L. V. Kantorovich on 3 VII 1965)

In the present note a number of theorems are given on subspaces and bases in spaces $C(T)$*, where T is a metric compactum, and also in Banach subspaces of a certain abstract class that includes the class of all spaces of the form $C(T)$. In this connection, the case of a subspace of C will be considered in greater detail. All spaces considered here are assumed to be real.

§ 1. We shall use the following terminology and notation.

1. Let $\{x_i\}_{i=1}^{\infty}$ be a sequence of pairwise distinct elements of a metric compactum T. A basis $\{f_i\}_{i=1}^{\infty}$ in $C(T)$ is called **interpolating with nodes** $\{x_i\}_{i=1}^{\infty}$ if, in the expansion of any $f \in C(T)$,

$$f = \sum_{i=1}^{\infty} a_i f_i,$$

the n-th partial sum $\sum_{i=1}^{n} a_i f_i$ interpolates $f(x)$ at the nodes $x_1, \dots, x_n$.

2. We shall denote by $\gamma_{\{e_i\}_{i=1}^{\infty}}$ the **index** of the sequence $\{e_i\}_{i=1}^{\infty}$ of elements of a Banach space (see the definition in (2) or (3)).

3. For isomorphic Banach spaces E_1 and E_2, we shall denote by $d(E_1, E_2)$ the quantity

$$d(E_1, E_2) = \inf_T \|T\| \cdot \|T^{-1}\|,$$

where T ranges over all isomorphisms of E_1 onto E_2 (the quantity $\ln d(E_1, E_2)$ was introduced by S. Banach and S. Mazur (see (4), p. 211)).

4. For a subspace P of a Banach space E, adopt the notation

$$\lambda(P, E) = \inf_A \|A\|,$$

where A ranges over all projection operators from E onto P. The projection constant of a Banach space R is the quantity

$$\lambda(R) = \sup_B \lambda(R, B),$$

where B ranges over all Banach spaces containing R as a subspace.

Theorem 1. Let $\{x_i\}_{i=1}^\infty$ be a sequence of pairwise distinct elements of a metric compactum T. In order that there exist in $C(T)$ an

---

* $C(T)$ (T a metric compactum) is the space of all continuous functions on T, with the naturally defined vector operations and the norm $\|f\| = \max |f(x)|$. If T is a collection of n points, then $C(T)$ is denoted by c^n, $n = 1, 2, \ldots$.

interpolating basis with nodes $\{x_i\}_{i=1}^\infty$, it is necessary and sufficient that $\{x_i\}_{i=1}^\infty$ be dense in T.

Theorem 2. Let $\{x_i\}_{i=1}^\infty$ be a sequence of pairwise distinct elements of the metric compactum T, dense in T. For any preassigned $\varepsilon > 0$, there exists in $C(T)$ an interpolating basis $\{e_i\}_{i=1}^\infty$ with nodes $\{x_i\}_{i=1}^\infty$ such that:

1) $\gamma_{\{e_i\}_{i=1}^\infty} < 1 - \varepsilon$;
2) $d(L_{1,n}, c^n) < 1+\varepsilon$, $\quad n = 1, 2, \ldots$, where $L_{1,n}$ is the linear span of $e_1, \ldots, e_n$.

Define the class $\mathfrak{C}$ of Banach spaces, assuming that $E \in \mathfrak{C}$ if, for every $\varepsilon > 0$ and every finite-dimensional subspace P of E, there exists a finite-dimensional subspace $Q = Q(P, \varepsilon)$ in E, $Q \supset P$, such that $d(Q, c^n) < 1+\varepsilon$, where $n = \dim Q$.

Theorem 3. If in a Banach space E, for arbitrary $\varepsilon > 0$ and finite-dimensional subspace P, there exists a finite-dimensional subspace $Q = Q(P, \varepsilon)$ such that the conditions

1) $d(Q, c^n) < 1 + \varepsilon$ ($n = \dim Q$);
2) $\displaystyle\sup_{x \in P,\ \|x\|=1} \inf_{y \in Q} \|x + y\| < \varepsilon$,

are satisfied, then $E \in \mathfrak{C}$.

It follows from Theorem 3, for example, that c_0 is a space of class $\mathfrak{C}$, since the linear span of the first n elements of the natural basis in c_0 is isometric to c^n (for the definition of the spaces c_0 and c, see ([4])).

Theorem 4. For any metric compactum T, $C(T) \in \mathfrak{C}$.

Theorem 5. For any metric compactum T, $d(c_0, C(T)) \geqslant 2$.

Theorem 5 means that $d(c_0, c) \geqslant 2$ and thereby, in particular, gives an answer to one question of S. Banach (([4]), p. 211). Theorems 4 and 5 mean, moreover, that the class of all spaces of the form $C(T)$ is a proper subset of the set of all separable spaces of class $\mathfrak{C}$.

For an arbitrary set $\mathfrak{S}$, denote by $m(\mathfrak{S})$ the space of all real bounded functions on $\mathfrak{S}$, with the naturally defined vector operations and the norm

$$\|f\| = \sup_{x\in\mathfrak{S}} |f(x)|.$$

Theorem 6. For any set $\mathfrak{S}$, $m(\mathfrak{S}) \in \mathfrak{C}$.

Theorem 7. In a separable space of class $\mathfrak{C}$ there exists a basis.

Each of Theorems 1, 2, and 7 is a generalization of a theorem of F. S. Vakher on the existence of a basis in the space $C(T)$, where T is a metric compactum ([5]).

Theorem 8. If E is an infinite-dimensional space of class $\mathfrak{C}$, then in E there exists an infinite-dimensional subspace E', not isomorphic to E*.

Theorem 9. If P is a finite-dimensional subspace of a space E of class $\mathfrak{C}$, then $\lambda(P, E) = \lambda(P)$.

§ 2. We present two theorems on subspaces in C, which consist of functions possessing certain differentiability properties (some results on basic sequences in C consisting of such functions were obtained in ([6])**).

Theorem 10. If all elements of a subspace E of the space C are functions differentiable on $[0, 1]$, then E is finite-dimensional.

Nevertheless, in C there exist subspaces of infinite dimension consisting of functions differentiable on $(0, 1)$ (and even analytic on $(0, 1)$). An example may be furnished by the closure in C of the linear

* All subspaces considered here are assumed to be closed.

** A sequence $\{e_i\}_{i=1}^{\infty}$ in a Banach space is called **basic** if it is a basis in the closure of its linear span.

envelopes of the sequence of powers $\{t^{n_k}\}_{k=1}^{\infty}$, $n_k > 0$,

$$\sum_{k=1}^{\infty} \frac{1}{n_k} < \infty \tag{7}$$

However, the class of such subspaces in C is very narrow, as the following shows.

Theorem 11. *If all elements of an infinite-dimensional subspace E of the space C are functions differentiable on $(0, 1)$, then for every $\varepsilon > 0$ there exists a subspace E_ε in E such that $d(E_\varepsilon, c_0) < 1 + \varepsilon$.*

Corollary. *If the elements of a reflexive subspace E of the space C are functions differentiable on $(0, 1)$, then E is finite-dimensional.*

Kharkov
Automobile and Highway Institute

Received
29 VI 1965

REFERENCES

1. Z. Semadeni, Bull. Acad. Polon. Sci., Ser. Sci. math., astr. et phys., **11**, 387 (1963).
2. M. M. Grinblyum, DAN, **31**, No. 5, 428 (1941).
3. V. I. Gurarii, DAN, **145**, No. 3, 504 (1962).
4. S. Banach, *A Course in Functional Analysis*, Kiev, 1948.
5. F. S. Vakher, DAN, **101**, No. 4, 539 (1955).
6. V. I. Gurarii, DAN, **148**, No. 3, 493 (1963).
7. A. F. Leont' ev, DAN, **72**, No. 4, 621 (1950).

Note: Figure translations are in progress. See original paper for figures.