ON AN ANALYTIC CONDITION ON THE RADICAL OF A COMMUTATIVE BANACH ALGEBRA AND RELATED QUESTIONS OF DECOMPOSABILITY

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Abstract

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ON AN ANALYTIC CONDITION ON THE RADICAL OF A COMMUTATIVE BANACH ALGEBRA AND RELATED QUESTIONS OF DECOMPOSABILITY

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The purpose of the present work is to establish a condition on the radical, "analytic" in its character, which ensures the decomposability of certain classes of commutative Banach algebras (c.b.a.) with such a radical. Let $\mathfrak{A}$ be a c.b.a.; $\mathfrak{R}$ its radical; A the quotient algebra $\mathfrak{A}/\mathfrak{R}$; and $P_{\mathfrak{A}}$ and P_A the sets of idempotents of the algebras $\mathfrak{A}$ and A. The natural epimorphism $\pi : \mathfrak{A} \to A$ induces a mapping $P_\pi : P_{\mathfrak{A}} \to P_A$, which, as shown in (^{1}), establishes a one-to-one correspondence between these two sets.

Bade and Curtis showed (^{2,3}) that questions of decomposability of a number of c.b.a.' s reduce to the following: will the preimage (under the mapping P_π) of every bounded subset of P_A be bounded in $\mathfrak{A}$? They also proved in (^{3}) that the latter is always the case if the radical $\mathfrak{R}$ is algebraically nilpotent, i.e. $\mathfrak{R}^n = 0$ for some natural n.

Definition. By the **radical sequence** of a c.b.a. $\mathfrak{A}$ with radical $\mathfrak{R}$ we shall mean the sequence

$$f(\mathfrak{A}, n) = \sup_{\substack{r \in \mathfrak{R} \\ \|r\|=1}} \|r^n\|.$$

Lemma 1. Let $\mathfrak{R}_1 \subseteq \mathfrak{R}$ be a closed ideal in $\mathfrak{A}$. Then

$$f(\mathfrak{A}/\mathfrak{R}_1, n) \leq f(\mathfrak{A}, n).$$

The proof follows immediately from the fact that the radical of the algebra $\mathfrak{A}/\mathfrak{R}_1$ is the image of the radical $\mathfrak{R}$ under the natural epimorphism of $\mathfrak{A}$ onto $\mathfrak{A}/\mathfrak{R}_1$, and from the definition of the norm in the quotient algebra.

Lemma 2. Let $f(n) > 0,\ n = 1, 2, \dots$, be a sequence such that

$$\lim_{n\to\infty} \sqrt[n^2]{f(n)} = 0;$$

let K be a constant. Then the set

$$\{M > 0 : \; M - M^n f(n) \le K^n; \; n = 1, 2, ...\}$$

is bounded.

Put $F(n) = f(n)^{-1/n}$; since

$$\lim_{n\to\infty} \sqrt[n]{F(n)} = \infty,$$

there is, evidently, a natural number N_0 such that $F(n) > K^{n+1} + 1$ for all $n \ge N_0$. Take $M > F(N_0)$; for some natural $N \ge N_0 + 1$,

$$F(N-1) < M \le F(N).$$

We have

$$M - M^N f(N) > F(N-1) - F(N)^N f(N).$$

But $F(N)^N f(N) = 1$, while $F(N-1) > K^N + 1$ in view of $N - 1 \ge N_0$. Therefore $M - M^N f(N) > K^N$. Consequently, the inequality $M - M^n f(n) \le K^n$ for $M > F(N_0)$ is not satisfied for all n, which proves the lemma.

The main result of the paper is the following.

Theorem 1. Let $\mathfrak{A}$ be a c.b.a., $\mathfrak{R}$ its radical, and suppose that the radical sequence satisfies the condition

$$\lim_{n\to\infty} \sqrt[n^2]{f(\mathfrak{A}, n)} = 0.$$

Let, further, a set $Q \subseteq P_{\mathfrak{A}}$ be such that its image πQ is bounded in $A = \mathfrak{A}/\mathfrak{R}$. Then Q is bounded in $\mathfrak{A}$.

For each $p \in P\mathfrak{A}$ put $\mathfrak{R}_p^0 = \{r \in \mathfrak{R} : \; pr = 0\}$ and $\mathfrak{R}_p^1 = \{r \in \mathfrak{R} : \; pr = r\}$. The $\mathfrak{R}_p^0$ and $\mathfrak{R}_p^1$ are closed ideals in $\mathfrak{A}$, $\mathfrak{R}_p^0 \mathfrak{R}_p^1 = 0$ and $\mathfrak{R} = \mathfrak{R}_p^0 \oplus \mathfrak{R}_p^1$.

We introduce the following notation: $\hat{\mathfrak{A}}_p^0 = \mathfrak{A}/\mathfrak{R}_p^0$; $\hat{\mathfrak{A}}_p^1 = \mathfrak{A}/\mathfrak{R}_p^1$; $\hat{\mathfrak{R}}_p^0$ and $\hat{\mathfrak{R}}_p^1$ are the radicals in $\hat{\mathfrak{A}}_p^0$ and $\hat{\mathfrak{A}}_p^1$; $\| \cdot \|_p^0$ and $\| \cdot \|_p^1$ are the norms in $\hat{\mathfrak{A}}_p^0$ and $\hat{\mathfrak{A}}_p^1$; $\pi_p^0 : \mathfrak{A} \to \mathfrak{A}/\mathfrak{R}_p^0$; $\pi_p^1 : \mathfrak{A} \to \mathfrak{A}/\mathfrak{R}_p^1$; $\hat{\pi}_p^0 : \hat{\mathfrak{A}}_p^0 \to \hat{\mathfrak{A}}_p^0/\hat{\mathfrak{R}}_p^0$ and $\hat{\pi}_p^1 : \hat{\mathfrak{A}}_p^1 \to \hat{\mathfrak{A}}_p^1/\hat{\mathfrak{R}}_p^1$ are the corresponding natural epimorphisms; finally, for convenience put $p_0 = \pi_p^0 p$, $p_1 = \pi_p^1 p$.

The following relations are obvious: $\hat{\mathfrak{R}}^0_p = \mathfrak{R}/\mathfrak{R}^0_p$; $\hat{\mathfrak{R}}^1_p = \mathfrak{R}/\mathfrak{R}^1_p$; $A = \mathfrak{A}/\mathfrak{R} = \hat{\mathfrak{A}}^0_p/\hat{\mathfrak{R}}^0_p = \hat{\mathfrak{A}}^1_p/\hat{\mathfrak{R}}^1_p$ and $\pi = \hat{\pi}^0_p\pi^0_p = \hat{\pi}^1_p\pi^1_p$. In addition, note that $p_1\hat{\mathfrak{R}}^1_p = 0$ (since $p\mathfrak{R} \subset \mathfrak{R}^1_p$).

We shall carry out the proof of the theorem in three stages.

1^0. The norms $\|p_1\|^1_p,\ p \in Q$, are bounded in the aggregate.

The set πQ, by the condition of the theorem, is bounded in A, i.e., for some constant K_1, $\|\pi p\| < K_1$ for $p \in Q$. But $\pi = \hat{\pi}^1_p\pi^1_p$; therefore from the definition of the norm in a quotient algebra it follows that for each $p \in Q$ there exist $\hat{p}_1 \in \hat{\mathfrak{A}}^1_p$ and $r^1_p \in \hat{\mathfrak{R}}^1_p$ such that $\hat{p}_1 = p_1 + r^1_p$, with $\|\hat{p}_1\|^1_p < K_1$. Clearly, in order to prove assertion 1^0, it suffices, in view of the triangle inequality, to establish the boundedness of the set $\{\|r^1_p\|^1_p;\ p \in Q\}$.

According to the multiplicative condition, $\|\hat{p}_1^n\|^1_p \leq (\|\hat{p}_1\|^1_p)^n < K_1^n$. On the other hand, in view of $r^1_p \in \hat{\mathfrak{R}}^1_p$ and $p_1\hat{\mathfrak{R}}^1_p = 0$, and also $p_1^n = p_1$, $\hat{p}_1^n = (p_1 + r^1_p)^n = p_1 + (r^1_p)^n = p_1 + r^1_p + (r^1_p)^n - r^1_p = \hat{p}_1 + (r^1_p)^n - r^1_p$. Applying the triangle inequality, we obtain $\|\hat{p}_1^n\|^1_p = \|r^1_p - (r^1_p)^n - \hat{p}_1\|^1_p \geq \|r^1_p\|^1_p - \|(r^1_p)^n\|^1_p - \|\hat{p}_1\|^1_p$, whence $\|r^1_p\|^1_p - \|(r^1_p)^n\|^1_p \leq K_1^n + K_1 < (K_1+1)^n$. But from the definition of the radical sequence it follows that $\|(r^1_p)^n\|^1_p \leq (\|r^1_p\|^1_p)^n f(\hat{\mathfrak{A}}^1_p, n)$, and by Lemma 1, $f(\hat{\mathfrak{A}}^1_p, n) \leq f(\mathfrak{A}, n)$. Thus,

$$\|r^1_p\|^1_p - \left(\|r^1_p\|^1_p\right)^n f(\mathfrak{A}, n) \leq (K_1 + 1)^n$$

simultaneously for all $p \in Q$ and $n = 1, 2, \ldots$. Consequently (Lemma 2), the norms $\|r^1_p\|^1_p,\ p \in Q$, are bounded in the aggregate, which, by the observation made above, proves assertion 1^0.

2^0. The norms $\|p_0\|^0_p,\ p \in Q$, are bounded in the aggregate. Let K_2 be a constant (which exists, as shown in 1^0) such that $\|p_1\|^1_p < K_2$ for $p \in Q$. Put $\bar{p} = e - p$ (e is the identity of the algebra $\mathfrak{A}$); obviously, without loss of generality one may assume that the set Q, together with each p, also contains $\bar{p}$ (the latter does not affect the boundedness of the set πQ). As is easily seen, $\mathfrak{R}^0_{\bar{p}} = \mathfrak{R}^1_p$ and $\mathfrak{R}^1_{\bar{p}} = \mathfrak{R}^0_p$. Therefore

$$\|p_0\|^0_p = \|\pi^0_p p\|^0_p = \|\pi^1_{\bar{p}} p\|^1_{\bar{p}} = \|\pi^1_{\bar{p}}(e - \bar{p})\|^1_{\bar{p}} \leq$$

$$\leq \|\pi^1_{\bar{p}} e\|^1_{\bar{p}} + \|\pi^1_{\bar{p}}\bar{p}\|^1_{\bar{p}} \leq \|e\| + \|\bar{p}_1\|^1_{\bar{p}} < \|e\| + K_2,$$

which proves assertion 2^0.

3^0. Completion of the proof. Let $K_3 = \|e\| + K_2$; in assertion 2^0 it was shown that $\|p_0\|^0_p < K_3$ for $p \in Q$. Consequently, for each $p \in Q$ there exists $r^0_p \in \mathfrak{R}^0_p$

such that $\|p + r_p^0\| < K_3$. But $p\mathfrak{R}_p^0 = 0$, whence also $pr_p^0 = 0$. Therefore, quite analogously to assertion 1^0, one can prove that

$$\|r_p^0\| - \|r_p^0\|^n f(\mathfrak{A}, n) \leq (K_3 + 1)^n, \qquad n = 1, 2, \ldots$$

Applying Lemma 2^0 once more, we obtain the boundedness of the set $\{\|r_p^0\| : p \in Q\}$, and with it, in view of the triangle inequality, of the set Q itself. Thus, the theorem is proved.

Remark. The conditions of Theorem 1 are satisfied, in particular, by the algebraically nilpotent radicals considered in (^{3}); in this case the radical sequence is, obviously, finite.

Combining Theorem 1 with the general results of (^{3}), we shall apply it to questions of decomposability of certain commutative Banach algebras. Recall that a commutative Banach algebra $\mathfrak{A}$ is called **strongly decomposable** if $\mathfrak{A} = B \oplus \mathfrak{R}$, where B is a closed subalgebra and $\mathfrak{R}$ is the radical of the algebra.

Theorem 2. *Let $\mathfrak{A}$ be a commutative Banach algebra with radical $\mathfrak{R}$ such that*

$$\lim_{n\to\infty} \sqrt[n^2]{f(\mathfrak{A}, n)} = 0.$$

Suppose further that $A = \mathfrak{A}/\mathfrak{R} = C(\Omega)$, where Ω is a totally disconnected compactum. Then $\mathfrak{A}$ is strongly decomposable.

Since $A = C(\Omega)$, the set P_A is bounded; consequently, by Theorem 1, the set $P_{\mathfrak{A}}$ in $\mathfrak{A}$ is also bounded. Applying Theorem 2.4 of (^{3}) (see also (^{4}), Corollary 2 to Theorem 3.10), we obtain the required result.

Theorem 3. *Let $\mathfrak{A}$ be a commutative Banach algebra with radical $\mathfrak{R}$ such that*

$$\lim_{n\to\infty} \sqrt[n^2]{f(\mathfrak{A}, n)} = 0.$$

Suppose further that $A = \mathfrak{A}/R = l_1$ (the algebra of absolutely convergent series with coordinatewise multiplication). Then $\mathfrak{A}$ is strongly decomposable.

Denote by Q the set of irreducible idempotents of the algebra $\mathfrak{A}$. Since πQ, the set of irreducible idempotents of the algebra l_1, is bounded, it follows, by Theorem 1, that Q is also bounded in $\mathfrak{A}$. To complete the proof it remains to apply Theorem 5.6 of (^{3}).

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Note: Figure translations are in progress. See original paper for figures.