ON THE REPRESENTATION OF STATES IN QUANTUM FIELD THEORY

MATHEMATICAL PHYSICS

1966

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UDC 530.145

MATHEMATICAL PHYSICS

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ON THE REPRESENTATION OF STATES IN QUANTUM FIELD THEORY

(Presented by Academician N. N. Bogolyubov on 16 XII 1965)

For the investigation of certain general properties of the S-matrix for arbitrary spin it is necessary to consider the problem of describing a dynamical system consisting of several noninteracting quantized fields characterized by operator-valued generalized functions $\hat{u}_1, \dots, \hat{u}_n$.

We introduce into consideration the countably normed Hilbert space (1)

$$D_S = \bigotimes_{n,l=0}^{\infty} D_S^{(n,l)}{}_{(R^4(n,l))} \tag{1}$$

of regular states $|\Phi\rangle$, where $D_S^{(n,l)}{}_{(R^4(n,l))}$ is the space of states of a bundle of noninteracting n particles and l antiparticles, and, for spins $s_r, j_t = 0, \frac{1}{2}, 1, \dots,\ r = 1, \dots, n,\ t = 1, \dots, l$, the space $D_S^{(n,l)}{}_{(R^4(n+l))}$ consists of elements of the space $S(R^{4(n+l)})$ of basic functions (2), represented in the form

$$\Phi^{(n,l)}_{(\sigma)n;\,(\nu)l}(p, q)_{n,l} = \bigotimes_{r=1}^{n} \mathcal{D}^{s_r}_{\sigma'_r \sigma_r}(l^{-1}(p_r)) \bigotimes_{t=1}^{l} \mathcal{D}^{j_t}_{\nu'_t \nu_t}(l^{-1}(q_t)) \hat{\Phi}^{(n,l)}_{(\sigma')n;\,(\nu')l}(p, a)_{n,l}, \tag{2}$$

where

$$\hat{\Phi}^{(n,l)}_{(\sigma)n;\,(\nu)l}(p, a)_{n,l} = \left(\bigotimes_{r=1}^{n} U(l(p_r)) \bigotimes_{t=1}^{l} U(l(q_t)) \Phi \right)^{(n,l)}_{(\sigma,\nu)n,l};$$

σ_r and ν_t are the spin projections, respectively, of a particle of mass m_r and spin s_r and of an antiparticle of mass μ_t and spin j_t on the third axis; p_r and q_t are their 4-energy-momenta with domain of definition

$$\Omega_{n,l} = \bigotimes_{r=1}^{n} \Omega_{m_r} \bigotimes_{t=1}^{l} \Omega_{\mu_t},$$

where Ω_m, $p_r^2 = m_r^2$, $p_{0r} > 0$, $r = 1, \ldots, n$, and Ω_{μ_t}: $q_t^2 = \mu_t^2$, $q_{0t} > 0$, $t = 1, \ldots, l$; $D_{\sigma'\sigma}(l(p))$ is a unitary irreducible representation of the $(2s+1)$-dimensional spin space (^{3}), $l(p) \in SL(2,c)$ and $l(p)\tilde{p} = p$, $\tilde{p} = (m, \vec{0})$, $(k)_f = (k_1, \ldots, k_f)$. We shall assume that $\Phi^{(n,l)} = 0$ for $n, l > \tilde{N}$, where $\tilde{N}$ is a sufficiently large number. A topology is introduced in D_S in a certain way.

The transformation property of the elements $|\Phi\rangle \in D_S$ with respect to the spinor Poincaré group $\widetilde{P}_+^{\uparrow}$ with elements (a, A), where a denotes 4-translations and A is an arbitrary element of $SL(2,c)$ corresponding to the element $\Lambda \in L_+^{\uparrow}$, is defined according to

$$U(a,A)|\Phi\rangle = \left\{ \exp\left[ia \left(\sum_r p_r + \sum_t q_t \right) \right] \times \right.$$
$$\left. \times \bigotimes_{r=1}^{n} \mathcal{D}^{s_r}_{\sigma'_r\sigma_r}(A) \bigotimes_{t=1}^{l} \mathcal{D}^{j_t}_{\nu'_t\nu_t}(A) \Phi^{(n,l)}_{(\sigma')n;\,(\nu')l}(A^{-1}p;\; A^{-1}q)_{n,l} \right\}. \tag{3}$$

The latter follows from the fact that

$$(U(a,A)\widehat{\Phi})^{n,l}_{(\sigma)_n;(\nu)_l}(p;q)_{n,l} = \exp\left[ia \left(\sum_r p_r + \sum_t q_t \right) \right] \times$$
$$\times \bigotimes_{r=1}^{n} \mathfrak{D}^{s_r}_{\sigma'_r\sigma_r}(R(A)p_r) \bigotimes_{t=1}^{l} \mathfrak{D}^{j_t}_{\nu'_t\nu_t}(R(A,q_t)) \widehat{\Phi}^{(n,l)}_{(\sigma')_n;(\nu')_l}(A^{-1}p; A^{-1}q)_{n,l}, \tag{4}$$

where $R(A,p) = l^{-1}(A^{-1}p)Al(p) \in SU(2,c)$.

In D_S the scalar product is introduced

$$\langle \Phi_1 \mid \Phi_2 \rangle = \sum_{n,l}^{\infty} \langle \Phi_1^{(n,l)} \mid \Phi_2^{(n,l)} \rangle < \infty, \tag{5}$$

where

$$\langle \Phi_1^{(n,l)} \mid \Phi_2^{(n,l)} \rangle = \langle (U(a,A)\Phi)^{(n,l)} \mid (U(a,A)\Phi_2)^{(n,l)} \rangle. \tag{6}$$

Let H_{L_2} be the Hilbert space obtained by completing D_S with respect to the nondegenerate scalar product (5). Then the family of spaces $D_S \subset H_{L_2} \subset D'_S$, where D'_S is the space of generalized states, forms a rigged physical Hilbert space (^{1},^{4}).

The space $D^{(1,0)'}_{S(R^4)}(m,s)$ of one-particle generalized states

$$\eta^{(1)}(\Phi^{(1)}) = \eta^{(1)}_{\sigma}(\Phi^{(1)}_{\sigma}) = \int \eta^{(1)}_{\sigma}(p)\Phi^{(1)}_{\sigma}(p)\frac{d^3p}{p}, \tag{7}$$

where $\Phi^{(1)}_{\sigma}(p) \in D^{(1,0)}_{S(R^4)}(m,s)$, is the space of irreducible representations of the spinor Poincaré group $\widetilde{P}_+\uparrow$. Generalized states are defined as common generalized eigenfunctions of the operators $P'_\mu, \vec{S}^{2'}, S'_3$ and form a canonical basis of the spinor space (^{5}). L. Schwartz' s kernel theorem (^{2}) makes it possible to define many-particle generalized states in the form

$$\rho^{(n)}_{(\sigma)_n}(\Phi^{(n)}_{(\sigma)_n}) = \eta^{(n)}_{(\sigma)_n}(\Phi^{(1)}_{\sigma_1}, \dots, \Phi^{(1)}_{\sigma_n}) = \eta^{(1)}_{\sigma_1}(\Phi^{(1)}_{\sigma_1}) \dots \eta^{(1)}_{\sigma_n}(\Phi^{n}_{\sigma_n}), \tag{8}$$

where $\Phi^{(n)}_{(\sigma)_n}(p)_n = \Phi^{(1)}_{\sigma_1}(p_1) \dots \Phi^{(1)}_{\sigma_n}(p_n)$. Therefore the vectors of an arbitrary generalized state $\mid \rho(\Phi)\rangle \in D'_S$ can be represented in the form

$$\mid \rho(\Phi)\rangle = \sum_{n,l\geq 0}{}' \int_{\Omega_{nl}} \cdots \int \rho^{(n,l)}_{(\sigma)_n;(\nu)_l}(p,q)_{n,l}\Phi^{(n,l)}_{(\sigma)_n;(\nu)_l}(p,q)_{n,l}\prod_{r=1}^{n}\frac{d^3p_r}{p_{0r}}\prod_{t=1}^{l}\frac{d^3q_t}{q_{0t}} \tag{9}$$

and possess the transformation property

$$U(a,A) \mid \rho(\Phi)\rangle = \{\rho^{(n,l)}_{(\sigma)_n;(\nu)_l}((U(a,A)\Phi)^{(n,l)}_{(\sigma)_n;(\nu)_l}(p,q)_{n,l})\}, \tag{10}$$

where $(U(a,A)\Phi)^{(n,l)}$ is defined according to (3).

In the space D'_S one can define weakly dense sets of vectors of generalized correlations of a special type. We shall present a scheme for constructing such a set.

To each function $f(x) \in S(R^4)$ we assign the Hermitian operator

$$A(f) = \int d^4x\, f(x)\, A(x) \text{ in } D_S, \qquad \langle\Psi \mid A(f) \mid \Phi\rangle, \quad \text{where } \Psi, \Phi \in D_S. \tag{11}$$

and we shall consider the quantity $A(f)$ as a linear continuous functional in $S(R^4)$ and call it an **operator-valued generalized function**.

Let us now consider the field operator $\chi(f)$ for the creation of a particle of mass $m \neq 0$ and spin s, defined as follows:

$$\chi(\tilde{f}) = \chi_\sigma(\tilde{f}^\sigma) = \int \frac{d^3p}{p_0}\, \tilde{f}^\sigma(p)\chi_\sigma(p), \quad \text{where} \quad \tilde{f}^\sigma(p) = \int e^{-ipx} f^\sigma(x)\, d^4x; \tag{12}$$

$$\chi(f)|\Phi\rangle = \left\{ \frac{1}{\sqrt{n}} \sum_{r=1}^{n} \sum_{\sigma_r = -s_r, \dots, s_r} \tilde{f}^{\sigma_r}(-p_r) \times \right.$$

$$\left. \times\, \Phi^{(n-1,l)}_{\sigma_1, \dots, \sigma_{r-1}, \sigma^{r+1}, \dots, \sigma_n; (\nu)_l}(p_1, \dots, p_{r-1}, p_{r+1}, \dots, p_n; (q)_l) \right\}; \tag{13}$$

$$U(a, A)\chi_\sigma(\tilde{f}^\sigma)U^{-1}(a, A) = \chi_\sigma\Big(e^{-iaAp}\mathcal{D}^s_{\sigma'\sigma}(R^{-1}(A, Ap))\tilde{f}^{\sigma'}(Ap)\Big) \tag{14}$$

for any $f^\sigma(x) \in S(R^4)$ and $|\Phi\rangle \in D_S$.

We construct the corresponding covariant operator by setting

$$a(\tilde{f}) = a_\alpha(\tilde{f}^\alpha) = \chi_\sigma\Big(\mathcal{D}^s_{\alpha\sigma}(l(p))\tilde{f}^\alpha(p)\Big) \tag{15}$$

with the transformation property

$$U(a, A)a_\alpha(\tilde{f}^\alpha)U^{-1}(a, A) = a_\alpha\Big(e^{-iaAp}\mathcal{D}^{s\alpha}_{\beta}(A^{-1})\tilde{f}^\beta(Ap)\Big). \tag{16}$$

Now the free field can be characterized by means of the covariant operator-valued generalized function $\varphi(f)$, defined in D_S:

$$\varphi(f)|\Phi\rangle = a(\tilde{f}(-p))|\Phi\rangle + \overset{*}{a}(\tilde{f}(p))|\Phi\rangle =$$

$$= \left\{ \frac{1}{\sqrt{n}} \sum_{r=1}^{n} \sum_{\sigma_r = -s_r, \dots, s_n} \mathcal{D}^{s_r}_{\alpha_r\sigma_r}(l(p_r))\tilde{f}^{\alpha_r}(-p_r) \times \right.$$

$$\times\, \Phi^{(n-1,l)}_{\sigma_1, \dots, \sigma_{r-1}, \sigma_{r+1}, \dots, \sigma_n; (\nu)_l}(p_1, \dots, p_{r-1}, p_{r+1}, \dots, p_n; (q)_l) +$$

$$\left. + \frac{1}{\sqrt{n+1}} \int \frac{d^3p}{p_0}\, \mathcal{D}^s_{\sigma\alpha}(l^{-1}(p))\tilde{f}^\alpha(p)\Phi^{(n+1,l)}_{\sigma, (\sigma)_n; (\nu)_l}(p, (p)_n; (q)_l) \right\}. \tag{17}$$

It is clear that

$$\varphi(f)D_S \subset D_S,$$

$$U(a,A)\varphi_\alpha(f^\alpha)U^{-1}(a,A) = \varphi_\alpha\left(\mathcal{D}^{s\alpha}_{\beta}(A^{-1})f^\beta(A^{-1}(x-a))\right) \tag{18}$$

for $f^\alpha(x) \in S(R^4)$.

Analogously, introducing the field operator in D_S according to

$$\chi(f) = \chi^{\dot\beta}(f_{\dot\beta}) = b^{\dot\beta}(\tilde f_{\dot\beta}) + (-1)^{2s}\overset{*}{b}{}^{\dot\beta}(\tilde f_{\dot\beta}), \tag{19}$$

where

$$b(\tilde f) = b^{\dot\beta}(\tilde f_{\dot\beta}) = \chi_\sigma\left(\overset{*}{\mathcal{D}}{}^{s\sigma}_{\dot\beta}(l(p))\tilde f_{\dot\beta}(p)\right), \tag{20}$$

$$U(a,A)b^{\dot\beta}(\tilde f_{\dot\beta})U^{-1}(a,A) = b^{\dot\beta}\left(e^{-iaAp}\overset{*}{\mathcal{D}}{}^{\dot\alpha}_{\dot\beta}(A)\tilde f_{\dot\alpha}(Ap)\right). \tag{21}$$

The field operators φ_α and $\chi^{\dot\beta}$ satisfy the Dirac equation

$$\varphi_\alpha(\Phi^\alpha) = \chi^{\dot\beta}(f_{\dot\beta}) \tag{22}$$

for arbitrary functions $\Phi^\alpha(x)$ and $f_{\dot\beta}(x)$ from $S(R^4)$ satisfying the relations

$$\begin{pmatrix} 0 & (-1)^{2s}\mathcal{D}^s(i\sigma\cdot\partial)_{\alpha\dot\beta} \\ (-1)^{2s}\mathcal{D}^s(i\tilde\sigma\cdot\partial)^{\dot\beta\alpha} & 0 \end{pmatrix}\begin{pmatrix}\Phi^\alpha(x)\\ f_{\dot\beta}(x)\end{pmatrix} = m^{2s}\begin{pmatrix}\Phi^\alpha(x)\\ f_{\dot\beta}(x)\end{pmatrix}, \tag{23}$$

where $\sigma = (\sigma_0, \vec\sigma)$, $\tilde\sigma = (\sigma_0, -\vec\sigma)$, $\vec\sigma = (\sigma_1, \sigma_2, \sigma_3)$ are the Pauli matrices, and σ_0 is the identity matrix.

Now suppose that there exists a set of nonproper field operators $\varphi^{(1)}_{\alpha_1}, \varphi^{(2)}_{\alpha_2}, \dots$, defined in D_S. Then, using nuclear

by theorem (?), we can construct a set of generalized-state vectors everywhere dense in D_S,

$$\{c_{n,l}\Omega^{*(n,l)}(f_{n,l})|0\rangle\}, \tag{24}$$

where $c_{n,l}$ are arbitrary complex numbers, $|0\rangle \in D_S$ are vacuum states, and

$$\Omega^{*(n,l)}_{(\alpha)n;(\beta)l}(f^{(\alpha)n;(\beta)l}_{n,l}) = a^{*(1)}_{\alpha_1}\dots a^{*(n)}_{\alpha_n}c^{*(1)}_{\beta_1}\dots c^{*(l)}_{\beta_l}(f^{(\alpha)n;(\beta)l}_{n,l}) \tag{25}$$

with the transformation property

$$U(a,A)\Omega^{*(n)}_{(\alpha)n}(f_n^{(\alpha)n})U^{-1}(a,A) =$$

$$= \Omega^{*(n)}_{(\alpha)n}\left(\exp\left[ia\sum_r Ap_r\right]\bigotimes_{r=1}^{n}\mathcal{D}^{s_r}_{\beta_r\alpha_r}(A^{-1})f_n^{(\beta)n}(Ap)_n\right) \tag{26}$$

for $\tilde{f}_m^{(\alpha)n}(p)_n \in S(R^{4n})$.

In the following work we shall study relativistically invariant matrix elements of the S-matrix, defined in the form:

$$S(\Phi^{(\alpha)m;(\beta)r}_{M,n}\Psi^{(\alpha)n;(\beta)l}_{n,l}) = \langle 0|\Omega^{(m,r)}_{(\alpha)m;(\beta)r}(\Phi^{(\alpha)m;(\beta)r}_{m,r})S\Omega^{*(n,l)}_{(\alpha)n;(\beta)l}(\Psi^{(\alpha)n;(\beta)l}_{n,l})|0\rangle. \tag{27}$$

Here we shall regard the S-matrix as a unitary operator defined in the equipped physical Hilbert space $D \subset H \subset D'$, which maps one generalized state (25) from D', corresponding to an incident wave, into some other generalized state from D', corresponding to an outgoing wave.

Taking this opportunity, I express my deep gratitude to N. N. Bogoliubov, Nguyen Van Hieu, and A. V. Efremov for their interest in the work and for valuable comments.

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for Nuclear Research

Received
7 XII 1965

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Note: Figure translations are in progress. See original paper for figures.