ON ONE METHOD FOR COMPUTING THE RADON–NIKODYM DERIVATIVE OF TWO GAUSSIAN MEASURES

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Abstract

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ON ONE METHOD FOR COMPUTING THE RADON–NIKODYM DERIVATIVE OF TWO GAUSSIAN MEASURES

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Let $\xi(t, \omega)$, $t \in [a, b]$, $\omega \in \Omega$, be a family of real-valued functions; B the smallest σ-algebra of subsets of Ω with respect to which the family $\xi(t)$, $t \in [a, b]$, is measurable; $\mathcal{P}_i$, $i = 1, 2$, measures on (Ω, B) with respect to which $\xi(t)$ is a Gaussian process with $M_i\xi(t) = 0$, $M_i\xi(t)\xi(s) = R_i(s, t)$, $i = 1, 2$. In this note, on the basis of the general theory of Aronszajn ([4]) of Hilbert spaces reproducing the kernel $R(s, t)$ (r.k.h.s. $R(s, t)$), a new formulation is given of necessary and sufficient conditions that the kernels $R_i(s, t)$ must satisfy in order that the measures $\mathcal{P}_1$ and $\mathcal{P}_2$ be equivalent (this circumstance will be denoted by the symbol $\mathcal{P}_1 \sim \mathcal{P}_2$), and a formula is indicated for the Radon–Nikodym derivative $d\mathcal{P}_2/d\mathcal{P}_1[\xi(t)]$ of the measures $\mathcal{P}_2$ and $\mathcal{P}_1$. The results obtained are related to a recent elegant result of Yu. Rozanov ([3]), concerning the case of Gaussian measures corresponding to stationary processes and formulated in spectral terms. They apply to the case when, with respect to both measures, the process $\xi(t)$ is a continuous-in-mean-square Gaussian Markov process (g.m. process). The latter situation was studied in ([12]) under additional restrictions concerning the regularity of the kernels $R_i(s, t)$. Further, the case is considered of Gaussian measures $\mathcal{P}_1$ and $\mathcal{P}_2$ in the space of functions $\xi(s, t)$ of two variables, where the measure $\mathcal{P}_1$ corresponds to a Wiener field (in the sense of ([13])), while $\mathcal{P}_2$ differs from $\mathcal{P}_1$ in its mean value or correlation function.

1. Denote by $H_i(\xi)$ the Hilbert spaces that are the linear closures of the random variables $\xi(t)$, $t \in [a, b]$, in the norm corresponding to the scalar product $\langle \gamma, \eta \rangle_i = M_i\gamma\eta$. For equivalence of the measures $\mathcal{P}_1$ and $\mathcal{P}_2$ it is necessary and sufficient that the symmetric operator S in $H_1(\xi)$, defined by the equality $M_2\gamma\eta = M_1\gamma S\eta$, satisfy the following conditions, established in ([2,6,10]).

Condition I. The operator S is bounded and has a bounded inverse.

Condition II. The operator $I - S$ is a Hilbert–Schmidt operator.

Let $H(R_i)$ be the r.k.h.s. $R_i(s, t)$, and let $\langle \cdot, \cdot \rangle_i$ be the scalar product in $H(R_i)$. The equality $M_1\xi(t)\gamma = \varphi(t)$ defines an isomorphic correspondence $\gamma \leftrightarrow \varphi(t)$

between $H_1(\xi)$ and $H(R_1)$. The operator S in $H_1(\xi)$ then corresponds to an operator $\overline{S}$ in $H(R_1)$ (defined by the relation $M_1\xi(t)S\gamma = \overline{S}\varphi(t)$, where $\gamma \leftrightarrow \varphi(t)$) such that $\overline{S}\varphi(t) = \langle R_2(s,t), \varphi(s)\rangle_1$; to the operator $I - S$ in $H_1(\xi)$ there corresponds the operator $\overline{I} - \overline{S}$ in $H(R_1)$ such that $(\overline{I} - \overline{S})\varphi(t) = \langle R_1(s,t) - R_2(s,t), \varphi(s)\rangle_1$. Conditions I and II, obviously, may also be referred to the operator $\overline{S}$. Let us find $d\mathcal{P}_2/d\mathcal{P}_1[\xi(t)]$. If $\mathcal{P}_1 \sim \mathcal{P}_2$, then the operator S has a discrete spectrum. Let x_k, $k = 1, 2, ...$, be a complete orthonormal system in $H_1(\xi)$ of eigenvectors of the operator S, corresponding to eigenvalues μ_k, $k = 1, 2, ...$ Then almost everywhere both with respect to $\mathcal{P}_1$ and with respect to $\mathcal{P}_2$,

$$\frac{d\mathcal{P}_2}{d\mathcal{P}_1}[\xi(t)] = \lim_{n\to\infty} \exp\frac{1}{2}\sum_1^n \left[\left(1 - \frac{1}{\mu_k}\right) - \ln\mu_k\right] \times$$

$$\times \exp\frac{1}{2}\sum_1^n \left[x_k^2\left(1 - \frac{1}{\mu_k}\right) - \left(1 - \frac{1}{\mu_k}\right)\right], \tag{1}$$

whereas $\mathcal{P}_1 \sim \mathcal{P}_2$ if and only if the limit on the right-hand side of (1) exists almost everywhere with respect to both measures (from this conditions I and II are obtained). It is easy to verify that $\mathcal{P}_1 \sim \mathcal{P}_2$ if and only if, both with respect to the measure $\mathcal{P}_1$ and with respect to the measure $\mathcal{P}_2$, there exists

$$\underset{n\to\infty}{\text{l. i. m.}}\ \frac{1}{2}\sum_1^n \left[x_k^2\left(1 - \frac{1}{\mu_k}\right) - \left(1 - \frac{1}{\mu_k}\right)\right] = F[\xi(t)].$$

Since, moreover, the series

$$\sum_1^\infty \left[\left(1 - \frac{1}{\mu_k}\right) - \ln\mu_k\right] = D$$

converges, it follows that

$$d\mathcal{P}_2/d\mathcal{P}_1[\xi(t)] = C\exp F[\xi(t)], \qquad C = \exp{}^{1/2}D.$$

This circumstance suggests the following construction.

Consider the random function $\zeta(s,t) = \xi(s)\xi(t) - R_1(s,t)$, $s, t \in [a,b]\times[a,b]$, and construct the Hilbert space $H_1(\zeta)$ corresponding to it. Denote $\Phi_1(s,t;u,v) = M_1\zeta(s,t)\zeta(u,v) = R_1(s,u)R_1(t,v) + R_1(s,v)R_1(t,u)$, and let $H(\Phi_1)$ be the r.k.H.s. of $\Phi_1(s,t;u,v)$, the scalar product in which we denote by $[\cdot,\cdot]_1$. Since $R_1(s,u)R_1(t,v)$ is the reproducing kernel of the direct product of the spaces $H(R_1) \otimes H(R_1)$ (see ([4])), we have

$$H(\Phi_1) \equiv [H(R_1) \otimes H(R_1)],$$

where $[H(R_1) \otimes H(R_1)]$ denotes the space of symmetric functions $f(s,t) = f(t,s)$ from $H(R_1) \otimes H(R_1)$ with scalar product

$$[\cdot,\cdot]_1 = \frac{1}{2}[\cdot,\cdot].$$

The correspondence $\gamma \leftrightarrow \varphi(s,t)$, where $M_1\zeta(s,t)\gamma = \varphi(s,t)$, defines an isomorphism between $H_1(\zeta)$ and $H(\Phi_1)$. Under conditions I and II the operator $\bar{I} - \bar{S}^{-1}$ is a symmetric Hilbert-Schmidt operator in $H(R_1)$ and, consequently, is given by a kernel $A(s,t) \in H(\Phi_1)$; moreover, since under condition I the operator $\bar{I} - \bar{S}^{-1}$ is a Hilbert-Schmidt operator if and only if the operator $\bar{I} - \bar{S}$ is such, the condition $A(s,t) \in H(\Phi_1)$ is equivalent to the condition

$$R_1(s,t) - R_2(s,t) \in H(R_1) \otimes H(R_1).$$

It is not difficult to check that $F[\xi(t)] \in H_1(\zeta)$ and that $\gamma \leftrightarrow A(s,t)$ under the isomorphism of $H_1(\zeta)$ and $[H(R_1) \otimes H(R_1)]$. Thus the following is true.

Theorem 1. $\mathcal{P}_1 \sim \mathcal{P}_2$ if and only if:

1) the equation $\langle R_2(s,t), \varphi(t)\rangle_1 = \mu\varphi(s)$ in $H(R_1)$ has no eigenvalue $\mu = 0$;
2) $R_1(s,t) - R_2(s,t) \in H(R_1) \otimes H(R_1)$.

In this case

$$d\mathcal{P}_2/d\mathcal{P}_1[\xi(t)] = C \exp\left\{ {}^{1/2}[A(s,t), \zeta(s,t)]_1 \right\}, \tag{2}$$

where $A(s,t)$ is the kernel of the operator $\bar{A} = \bar{I} - \bar{S}^{-1}$ in $H(R_1)$, defined by the identity

$$\langle \varphi, \psi\rangle_1 - \langle \varphi, \psi\rangle_2 = \langle \varphi, \bar{A}\psi\rangle_1;$$

C is the constant defined above through the eigenvalues of the operator $\bar{S}$.

Remark. In the case when $M_1\xi(t) = 0$, $M_2\xi(t) = m(t)$,

$$M_i[\xi(t) - M_i\xi(t)][\xi(s) - M_i\xi(s)] = R(s,t),$$

the conditions under which $\mathcal{P}_1 \sim \mathcal{P}_2$ and the corresponding formula for $d\mathcal{P}_2/d\mathcal{P}_1[\xi(t)]$ in terms of the r.k.H.s. $R(s,t)$, assertions related to Theorem 1, were obtained by Parzen ([8]) and by Gack ([5]). The scalar product $[A(s,t), \zeta(s,t)]_1$ in formula (2) has the same meaning as the scalar product appearing in the formulas for $d\mathcal{P}_2/d\mathcal{P}_1[\xi(t)]$ in ([8,5]). In the case when, under the measure $\mathcal{P}_1$, the process is a Wiener process, the definition of $[A(s,t), \zeta(s,t)]_1$ is equivalent to the definition of the multiple stochastic integral of Itô (([7]), cf. ([11])). In the particular case when $R_2(s,t) - R_1(s,t)$ is a positive definite kernel, condition 2) (together with another, less general formula for $d\mathcal{P}_2/d\mathcal{P}_1[\xi(t)]$) was indicated by Parzen in ([9]).

2. Let $\xi(t)$, $t \in [a,b]$, with respect to the measure $\mathcal{P}_1$ be a g.m. process, $M_1\xi(t) = 0$. The set of points of nondegeneracy of the process $\{t : D_1\xi(t) \neq 0\}$ is the union of a countable number of intervals, on each of which $M_1\xi(t)\xi(s) = R_1(s,t)$ is represented in the form $R_1(s,t) =$

$$= \varphi(s)\varphi(t)\min[\psi(s)/\varphi(s), \psi(t)/\varphi(t)],$$

where $\psi(t)$ and $\varphi(t)$ are continuous functions such that, within the interval under consideration, $\varphi(t) > 0$, $\psi(t) > 0$, and $\psi(t)/\varphi(t)$ is nondecreasing. Since the values of the g.m. process $\xi(t)$ on different intervals of nondegeneracy are mutually independent, we may restrict ourselves to considering one such interval (a,b).

Let, with respect to the measure $\mathcal{P}_2$, the process $\xi(t)$, $t \in [a,b]$, be a g.m. process and

$$M_2\xi(t) = 0, \qquad M_2\xi(t)\xi(s) = R_2(s,t) = \theta(s)\theta(t)\min[\rho(s)/\theta(s), \rho(t)/\theta(t)],$$

where the functions $\rho(t)$, $\theta(t)$ have the same properties as $\psi(t)$, $\varphi(t)$. Let $g(u)$, $a_1 = \psi(a)/\varphi(a) \leq u \leq \psi(b)/\varphi(b) = b_1$, be the function inverse to $\psi(t)/\varphi(t)$, i.e. $g[\psi(t)/\varphi(t)] = t$. Introduce the notation:

$$\xi[g(u)]/\varphi[g(u)] = \zeta(u), \qquad \rho[g(u)]/\varphi[g(u)] = \mu(u), \qquad \theta[g(u)]/\varphi[g(u)] = \nu(u),$$

$$A(u,v) = \int_{a_1}^{b_1} \frac{\varphi'(u)}{\varphi(u)}\, e(s,u)\min(t,u)\,du + \int_{a_1}^{b_1} \frac{\varphi'(u)}{\varphi(u)}\, e(t,u)\min(s,u)\,du - \int_{a_1}^{b_1} \left[\frac{\varphi'(u)}{\varphi(u)}\right]^2 \min(s,u)\min(t,u)\,du,$$

Theorem 2. $\mathcal{P}_1 \sim \mathcal{P}_2$ if and only if:

1) $\mu(a_1) = 0$, if $a_1 = 0$, and $\mu(a_1) \neq 0$, if $a_1 \neq 0$; $\mu(u) > 0$ for $u > a_1$, $\nu(u) > 0$ for $u \geq a_1$;

2) The function $\mu(u)$ is absolutely continuous in u, the function $\nu(u)$ is absolutely continuous in u, and for $u > a_1$,

$$\nu^2(u)\int_{a_1}^{u} \mu'^2(u)\,du + \mu^2(u)\int_{u}^{b_1} \nu'^2(u)\,du < \infty$$

for every fixed $u \in [a_1, b_1]$;

3)
$$\mu'(u)\nu(u) - \mu(u)\nu'(u) \equiv 1;$$

4)
$$\int_{a_1}^{b_1} \nu'^2(u)\left(\int_{a_1}^{u} \mu'^2(v)\,dv\right)du < \infty.$$

In this case

$$\frac{d\mathcal{P}_2}{d\mathcal{P}_1}[\xi(t)] = C \exp \frac{1}{2} \left[\int_{a_1}^{b_1} \int_{a_1}^{b_1} \frac{\partial^2}{\partial u \, \partial v} A(u,v) \, d\xi(u) \, d\xi(u) + 2 \frac{\xi(a_1)}{a_1} \int_{a_1}^{b_1} \frac{\partial}{\partial u} A(a_1, u) \, d\xi(u) + \frac{[\xi^2(a_2) - a_1]}{a_1^2} A(a_1, a \right. \tag{3}$$

where

$$C = \exp \left\{ -\frac{1}{2} \int_{a_1}^{b_1} u \left[\frac{\nu'(u)}{\nu(u)} \right]^2 du \right\}$$

and the multiple stochastic integral is defined in the sense of Itô. If, in addition, the condition is satisfied:

5) $\lim\limits_{u \to b_1} \nu(u)$ exists and is not equal to 0,

then

$$\frac{d\mathcal{P}_2}{d\mathcal{P}_1}[\xi(t)] = \left[\frac{\nu(b_1)}{\nu(a_1)} \right]^{1/2} \exp \left\{ \frac{1}{2} \left[\xi^2(a_1) \left(\frac{1}{a_1} - \frac{1}{\nu(a_1)\mu(a_1)} \right) + \int_{a_1}^{b_1} \nu'(u) \, d \frac{\xi^2(u)}{\nu(u)} \right] \right\}; \tag{4}$$

if $a_1 = 0$, then in (3) and (4) we put

$$\frac{\xi(a_1)}{a_1} \int_{a_1}^{b_1} \frac{\partial}{\partial u} A(a_1, u) \, d\xi(u) = \frac{\xi^2(a_1) - a_1}{a_1^2} A(a_1, a_1) = \xi^2(a_1) \left(\frac{1}{a_1} - \frac{1}{\nu(a_1)\mu(a_1)} \right) = 0.$$

This theorem contains, as a special case, the main result of the paper ([12]).

3. Let the random function $\xi(s,t) = \xi(P)$, $P = (s,t) \in [0,1] \times [0,1]$, be, with respect to the Gaussian measure $\mathcal{P}_1$, a Wiener field in the sense-

(13), i.e. $M_1 \xi(s,t) = 0$, $M_1 \xi(s,t)\xi(u,v) = \min(s,u)\min(t,v)$, while $\mathcal{P}_2$ is a Gaussian measure such that $M_2 \xi(s,t) = m(s,t)$, $M_2[\xi(s,t) - m(s,t)][\xi(u,v) - m(u,v)] = \min(s,u)\min(t,v)$ in Theorem 3, or $M_2 \xi(s,t) = 0$, $M_2 \xi(s,t)\xi(u,v) = R(s,t;u,v)$ in Theorem 4. Denote

$$\int_0^s \int_0^t b(s',t') \, ds' \, dt' = \int_0^P b(P') \, dP', \quad \min(s,u)\min(t,v) = \min(P,Q),$$

and so on.

Theorem 3. $\mathcal{P}_1 \sim \mathcal{P}_2$ if and only if

$$m(P) = \int_0^P b(P') \, dP',$$

where

$$\int_0^1 [b(P)]^2 \, dP < \infty.$$

Moreover,

$$\frac{d\mathcal{P}_2}{d\mathcal{P}_1}[\xi(P)] = \exp\left\{-\frac{1}{2}\int_0^1 [b(P)]^2\,dP + \int_0^1 b(P)\,d\xi(P)\right\}.$$

Theorem 4. $\mathcal{P}_1 \sim \mathcal{P}_2$ if and only if:

1) the function $R(P;Q)$ can be represented in the form

$$R(P;Q) = \min(P;Q) + \int_0^P\int_0^Q b(P';Q')\,dP'\,dQ',$$

where

$$\int_0^1\int_0^1 [b(P;Q)]^2\,dP\,dQ < \infty;$$

2) the equation

$$\int_0^1 b(P;Q)f(Q)\,dQ = \lambda f(P)$$

has no eigenvalue -1 in L^2.

In this case

$$\frac{d\mathcal{P}_2}{d\mathcal{P}_1}[\xi(P)] = C\exp\left\{\frac{1}{2}\int_0^1\int_0^1 A(P;Q)\,d\xi(P)\,d\xi(Q)\right\},$$

where the stochastic integral is understood in Itô' s sense, and the function $A(P;Q)$ is determined from the condition that

$$f(P) = g(P) - \int_0^1 A(P;Q)g(Q)\,dQ$$

is a solution in L^2 of the equation

$$f(P) + \int_0^1 b(P;Q)f(Q)\,dQ = g(P).$$

The constant C can be determined through the eigenvalues $\lambda_k,\ k = 1, 2, \dots,$ of the equation

$$\int_0^1 b(P;Q)f(Q)\,dQ = \lambda f(P)$$

by the formula given in item 1, putting $\mu_k = \lambda_k + 1$.

The proof of these theorems is based on the result of Parzen–Hájek ([5,8]), which was discussed in the remark after Theorem 1, and on Theorem 1, whose reformulation for the case of fields $\xi(s,t)$ is obvious.

Theorem 3 generalizes Theorem 3 of paper ([13]), and Theorem 4 is similar to results of Shepp ([11]) and of the author ([1]), pertaining to the one-dimensional case.

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Note: Figure translations are in progress. See original paper for figures.