Uniqueness of solutions of inverse problems for metaharmonic potentials

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Abstract

Full Text

Preamble

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On the Uniqueness of Solutions to Inverse Problems of Metaharmonic Potentials

The so-called inverse problems of potential theory belong to the class of problems in mathematical physics that are ill-posed in the classical sense. Nevertheless, their applied significance is so substantial that they have recently become prominent among the relevant problems of modern mathematical analysis [?, ?, ?, ?]. Questions regarding the uniqueness of solutions to these problems are of critical importance in this direction. In this paper, we investigate the uniqueness of solutions for two inverse problems of metaharmonic potentials ($\varkappa > 0$) [?, ?, ?], which include the Newtonian potential as a special case when $\varkappa = 0$.

We introduce the following notation: E_n is the n-dimensional Euclidean space; $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_n)$ are points in this space; $r_{xy} = |y - x|$ is the distance between x and y; Γ_A is the boundary of an open set $A \subset E_n$; $U(x)$ is a function that is metaharmonic in a domain, i.e., a regular solution to the metaharmonic equation:

$$\Delta U = \varkappa^2 U \quad (\varkappa = \text{const} > 0); \tag{0.1}$$

$K(x, y)$ is the fundamental solution of equation (0.1) that vanishes at infinity, for example:

$$K(x, y) = \frac{e^{-\varkappa r_{xy}}}{4\pi r_{xy}} \quad \text{for } n = 3, \varkappa > 0;$$

$$K(x,y)=\frac{1}{2\pi}K_0(\varkappa r)\quad \text{for } n=2,\varkappa>0;$$

where the function $K_0(t)$ has a logarithmic singularity at $t=0$ and decreases exponentially at infinity; $K(x,y)=\frac{1}{2\pi}\ln\frac{1}{r_{xy}}$ for $\varkappa=0,n=2$. We define the function

$$V(x)=\int_D \mu(y)K(x,y)dy, \tag{0.2}$$

where $K(x,y)$ is the fundamental solution of equation (0.1), as the metaharmonic (volume) potential of the domain D with density $\mu(y)$. If the case $\varkappa=0,n>2$ is considered separately, we refer to the function $V(x)$ as the Newtonian potential. Throughout the following, unless otherwise specified, we assume that the domain D is, generally speaking, multiply connected and, furthermore, that the density $\mu(y)>0$ almost everywhere for points $y\in D$. We now provide the formulation of the problems considered in this work.

Problem 1. Let D_1 and D_2 be finite domains (or open sets), and let $V(x)$ be a function defined throughout the space, representing the potential determined by a body with a given density $\rho(y)$. The objective is to determine the relative positioning of these domains under the condition that their external potentials ($\varkappa>0$) are equal; that is, $V_1(x)=V_2(x)$ for $x\in E_n\setminus(\bar D_1\cup\bar D_2)$.

For the case of the Newtonian potential ($\varkappa=0$), the question of the uniqueness of the solution to Problem 1 has been investigated in several works, including [**?**, **?**, **?**, **?**, **?**, **?**]. A more detailed bibliography for this specific case can also be found in the author' s previous work [**?**]. The current study addresses the case where $\varkappa>0$ for this problem. This problem was previously addressed by the author in works [**?**, **?**, **?**].

Problem 2. Let S_α ($\alpha=1,2$) be the boundary of a domain.

$$V(x)=\int_S \nu(y)K(x,y)d\sigma_y,\quad (\varkappa>0)$$

is a metaharmonic simple layer potential with a given density. The objective is to determine the relative positioning of the surfaces S_α under the condition that the potentials satisfy the relation:

$$V_{S_1}(x)=V_{S_2}(x)\quad \text{for } x\in E_n\setminus(\bar D_1\cup\bar D_2).$$

For the logarithmic simple layer potential, this problem was investigated in the works of I. M. Rapoport [**?**] and A. A. Mikheeva [**?**]. It should be noted, however, that these works do not provide uniqueness theorems. In the first section of the present paper (Lemma 1), we prove the uniqueness of the representation of a certain functional as the sum of an external volume potential and simple and double layer potentials. In the second section, Theorems 1 and 2 investigate the uniqueness of the solution to Problem 1 for variable density. We note that in the case where $\varkappa = 0$, Theorems 1 and 2 yield new uniqueness theorems for the Newtonian potential, which specifically generalize the results found in [?, ?, ?, ?]. The question regarding the uniqueness of the solution to Problem 2 is also investigated in Section 3 (Theorems 4, 5) for constant density under various constraints. In particular, for the case where $\varkappa = 0$, Theorem 5 implies uniqueness theorems in n-dimensional space for the Newtonian potential of a simple layer.

1. Fundamental Identity for Volume, Simple Layer, and Double Layer Potentials

Let D_a (where $a = 1, 2$) be finite domains bounded by surfaces S_a, such that S_a is the boundary of the set D_a. Furthermore, we assume throughout the following discussion that each boundary S_a $(a = 1, 2)$ belongs to the class $L(1, \lambda)$ (see [?]). We introduce the following notation:

$$\phi(x) = \int_D \rho(y) K(x, y) dy \tag{1.1}$$

$$\psi(x) = \int_\Gamma \mu(y) K(x, y) d\sigma_y \tag{1.2}$$

We refer to the function (1.1) as a metaharmonic volume potential, and the function (1.2) as a simple layer potential. Let α and β be real numbers. We denote:

$$\Phi(x) = \alpha\phi(x) + \beta\psi(x) \tag{1.3}$$

If we consider open bounded sets A_α $(\alpha = 1, 2)$, each consisting of a finite number of domains:

$$A_\alpha = \bigcup_{i=1}^{n_\alpha} T_i^\alpha, \quad \alpha = 1, 2 \tag{1.5}$$

where n_α are fixed numbers, and Γ_α $(\alpha = 1, 2)$ denotes the boundary of the set A_α, then throughout equations (1.1-1.4), we replace T_α with A_α accordingly.

Lemma 1. If the bounded functions $\Phi_\alpha(x)$ $(\alpha = 1, 2)$ are such that the equality

$$\Phi_1(x) = \Phi_2(x) \quad \text{for } x \in E_n \setminus (A_1 \cup A_2) \tag{1.6}$$

holds, then for any metaharmonic $(\varkappa > 0)$ function $U(y)$ defined in the domain $A_1 \cup A_2$, the following equality holds:

$$J(U) = P(U). \tag{1.7}$$

Proof. We construct two domains D_k $(k = 1, 2)$ with analytic boundaries such that, according to the well-known formula for any metaharmonic function $U(x)$ in the domain D_k, the following relation holds for points $y \in D_k$:

$$U(y) = \int_{L_k} \left(U(x) \frac{\partial K(x, y)}{\partial n_x} - K(x, y) \frac{\partial U(x)}{\partial n_x} \right) dL_x \tag{1.8}$$

where n_x is the outward normal to the surface L_k. By multiplying expression (1.8) by the function $\rho_a(y)$ $(a = 1, 2)$ and integrating over the set $D_{a'}$ $(a' = 1, 2)$ respectively, we obtain:

$$\int_{D_{a'}} \rho_a(y)U(y)dy = \int_{D_{a'}} \rho_a(y) \left[\int_{L_k} \left(U(x)\frac{\partial K(x,y)}{\partial n_x} - K(x,y)\frac{\partial U(x)}{\partial n_x}\right) dL_x\right] dy \tag{1.9}$$

Furthermore, by multiplying expression (1.8) by the function $\mu(y)$ and integrating over the boundary Γ, we have:

$$\int_{\Gamma} \mu(y)U(y)d\sigma_y = \int_{\Gamma} \mu(y) \left[\int_{L_k} \left(U(x)\frac{\partial K(x,y)}{\partial n_x} - K(x,y)\frac{\partial U(x)}{\partial n_x}\right) dL_x\right] d\sigma_y \tag{1.10}$$

By multiplying (1.9) and (1.10) by the appropriate coefficients and summing the resulting terms, we arrive at the final expression (1.11). According to the conditions of the lemma, since (1.6) holds, it follows from the well-known properties of potentials that the limit values coincide (1.12). Therefore, the assertion of Lemma (1.7) follows directly from (1.12), (1.6), and (1.11).

Before formulating Lemma 2, we introduce the concept of a generalized solution, following the methodology established in works [**?**, **?**] (see also [**?**]). In the following discussion, we assume that the boundary of the domain D (which may be multiply connected) has no interior points. If the Dirichlet problem for the metaharmonic equation (0.1) is solvable in a given domain, we refer to it as a normal domain.

Definition. The function $U_f(x)$ constructed by the method described above is called the generalized solution of the Dirichlet problem for the metaharmonic equation (0.1) in the domain D with continuous boundary data $f(x)$.

Lemma 2. If the bounded functions are such that the external potentials coincide, then for any generalized solution U of the metaharmonic equation (0.1) for the Dirichlet problem in domain T with boundary data f continuous on the boundary S, the fundamental identity holds. The proof of this lemma is conducted analogously to Lemma 2 in reference [**?**].

2. Uniqueness Theorems for Problem 1

Let T_1, T_2 be finite domains satisfying the conditions of Section 1. Henceforth, we shall adhere to the notation established in [**?**]. Let S denote the boundary of the set $T_1 \cup T_2$. We also introduce the following notation:

$$S_1 = S \setminus \Gamma_2, \quad S_2 = S \setminus \Gamma_1 \tag{2.1}$$

Theorem 1. Assume there exists at least one constant vector $q = (q_1, \dots, q_n)$ such that: 1) Every line parallel to the vector q intersects each of the sets T_i either at two points or along two segments; 2) For a positive function $\mu(x) \in C$, the condition $\frac{\partial\mu}{\partial q} \geq 0$ is satisfied; 3) For the domains T_i with a given density, the following equality holds:

$$V_{T_1}(x) = V_{T_2}(x) \quad \text{for } x \in E_n \setminus (T_1 \cup T_2),$$

then $T_1 = T_2$.

Proof. By choosing coordinates such that the direction of the y-axis coincides with the vector q and applying Lemma 1 to condition 3, it follows that:

$$\int_{T_1} \mu(y)U(y)dy = \int_{T_2} \mu(y)U(y)dy \tag{2.2}$$

for any metaharmonic function $U(y)$ in the domain $T_1 \cup T_2$. We choose a function of the form:

$$U(y) = \frac{\partial H}{\partial q} \tag{2.3}$$

where H is a metaharmonic function. Substituting (2.3) into (2.2) and utilizing the second condition of the theorem, we transform the volume integrals into surface integrals to obtain:

$$\int_{S_1} H(y)\mu(q, n_1)d\sigma = \int_{S_2} H(y)\mu(q, n_2)d\sigma \tag{2.4}$$

where (q, n_i) denotes the scalar product of the vector q and the unit outward normal vector to the surface S_i. We define a function on the surface as follows: 1 for S_1, and 0 for the remaining boundary. As in the proof of Theorem 1 [**?**], the equality (2.4) can be extended to the solution of the Dirichlet problem for equation (0.1). From (2.5) and (2.6), we obtain the following equality:

$$\Phi(q) = 0 \tag{2.7}$$

Repeating the reasoning from the proof of the theorem, it is easy to verify that all integrals are non-negative and at least one is strictly greater than zero. This leads to a contradiction, proving the theorem.

Theorem 2. There exists at least one constant vector q such that the following conditions are satisfied: 1) Each of the sets $S_\alpha \setminus (S_1 \cap S_2)$ is non-empty, and any line parallel to the vector q intersects $\Gamma_1 \cup \Gamma_2$ at no more than two points or two segments; 2) For a positive function $\mu(x)$ on $S_1 \cup S_2$, the condition $\frac{\partial\mu}{\partial q} \geq 0$ is satisfied; 3) For the sets S_1, S_2 with a given density μ, the equality $V_1(x) = V_2(x)$ holds. Then $S_1 = S_2$.

The proof of the theorem is analogous to that for the Newtonian potential (see the theorem in [**?**]).

3. Uniqueness Theorems for the Simple Layer Potential

Let $V(x)$ denote the simple layer potential with density μ for equation (0.1):

$$V(x) = \int_S \mu K(x,y) d\sigma_y \tag{3.1}$$

Theorem 4. If for surfaces S_1 and S_2 the following relation holds:

$$\text{mes } S_1 = \text{mes } S_2 \tag{3.2}$$

and for the metaharmonic simple layer potential of constant density μ, the equality

$$V_1(x) = V_2(x) \quad \text{for } x \in E_n \setminus (\bar{D}_1 \cup \bar{D}_2) \tag{3.3}$$

holds, then $S_1 = S_2$.

Proof. We proceed by contradiction. Based on Lemma 1, it follows from condition (3.3) that:

$$\int_{S_1} u\, d\sigma = \int_{S_2} u\, d\sigma \tag{3.4}$$

for any metaharmonic function u in the domain D. Define a function f on the surface $S = S_1 \cup S_2$ as follows:

$$f = \begin{cases} 1 & \text{for } x \in S_1 \setminus S_2 \\ -1 & \text{for } x \in S_2 \setminus S_1 \end{cases} \tag{3.6}$$

As before, we extend equality (3.4) to a function u that is a generalized solution taking values (3.6) almost everywhere. Consequently, we have:

$$\int_{S_1 \setminus S_2} u\, d\sigma = \int_{S_2 \setminus S_1} u\, d\sigma \tag{3.7}$$

Substituting the data from (3.6) into (3.7), we obtain:

$$\int_{S_1 \setminus S_2} 1\, d\sigma + \int_{S_2 \setminus S_1} 1\, d\sigma = \int_{S_1 \setminus S_2} (1-u)\, d\sigma + \int_{S_2 \setminus S_1} (1+u)\, d\sigma \tag{3.8}$$

Based on the extremum principle, $|u| < 1$; therefore, from (3.8) it follows that $\text{mes}(S_1 \setminus S_2) + \text{mes}(S_2 \setminus S_1) < \text{mes}(S_1 \setminus S_2) + \text{mes}(S_2 \setminus S_1)$, which contradicts the condition of Theorem (3.2). The theorem is proved.

Theorem 5. If for the boundaries Γ_a $(a = 1, 2)$ the relation mes Γ_1 = mes Γ_2 holds and the simple layer potentials with constant density are equal, then $\Gamma_1 = \Gamma_2$.

The proof is carried out similarly to Theorem 4, utilizing the construction for the set A^* as established in Theorem 4 [**?**].

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Note: Figure translations are in progress. See original paper for figures.