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Abstract

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MATHEMATICS

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NONTRIVIALITY CONDITIONS FOR CERTAIN CLASSES OF FUNCTIONS ANALYTIC IN AN ANGLE, AND QUASIANALYTICITY PROBLEMS

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1°. Let G be a connected set of points of the complex plane or, more generally, of some Riemann surface, and let $\{A_0 = 1, A_1, A_2, \dots\}$ be a given sequence of nonnegative numbers (some of which may be equal to $+\infty$). We shall denote by $D\{G; A_n\}$ the class of functions $f(z)$ ($z \in G$), infinitely differentiable on G , for which

$$\sup_{z \in G} |f^{(n)}(z)| \leq CA_n \quad (n = 0, 1, 2, \dots) \quad (1)$$

(the constant C depends on f). Obviously, the functions $f(z) \equiv \text{const}$ belong to any class $D\{G; A_n\}$; if, apart from them, the given class contains not a single function, we shall call it **trivial**. We pose the following problem:

A. Given a set G ; find necessary and sufficient conditions which the sequence $\{A_n\}_0^\infty$ must satisfy in order that the class $D\{G; A_n\}$ be nontrivial.

As an example, we point out that if G is the entire complex plane K (without the point at infinity), then any class $D\{K; A_n\}$ is trivial (Liouville's theorem); an analogous result holds when G is the Riemann surface of the function $\log z$. At the other extreme, when G is a bounded set, the necessary and sufficient condition for nontriviality of the class $D\{G; A_n\}$ is $A_1 > 0$.

In the present paper, problem A is solved for the case in which G is an angle in the complex plane or, more generally, on the Riemann surface of the function $\log z$. In doing so, a connection is revealed between problem A and the classical problem of quasianalyticity, and with an analogous problem in the complex domain.

2°. Definitions.

- 1) I_α ($0 \leq \alpha < \infty$) denotes the angle on the Riemann surface of the function $\log z$ defined by the inequalities $0 \leq |z| < \infty$, $-\pi\alpha/2 \leq \arg z \leq \pi\alpha/2$ (for $\alpha < 2$, instead of the Riemann surface one may consider the ordinary complex plane); γ_α denotes the boundary of I_α . Thus $I_0 = \gamma_0 = [0, \infty]$, and γ_1 is the imaginary axis.
- 2) The classes $D\{I_\alpha; A_n\}$ are denoted by $D_\alpha\{A_n\}$ ($0 \leq \alpha < \infty$); the classes $D\{\gamma_\alpha; A_n\}$ are denoted by $\widetilde{D}_\alpha\{A_n\}$ ($0 < \alpha \leq 1$).
- 3) The class $D_\alpha\{A_n\}$ ($0 \leq \alpha < \infty$) is called quasianalytic at the point 0 if there does not exist a function $f(z) \not\equiv 0$ belonging to this class and such that $f^{(n)}(0) = 0$ ($n = 0, 1, \dots$).

Theorem 1. Let $0 \leq \alpha \leq 1$. For the nontriviality of the class $D_\alpha\{A_n\}$ it is necessary and sufficient that

$$\inf_{n>0} (n^{1-\alpha} A_n^{1/n}) > 0. \quad (2)$$

This same condition is necessary and sufficient also for the nontriviality of the class $\widetilde{D}_\alpha\{A_n\}$.

Theorem 2. Let $\alpha > 1$. For the nontriviality of the class $D_\alpha\{A_n\}$ it is necessary and sufficient that

$$\int_0^\infty r^{-\alpha/(\alpha-1)} \log T(r) dr < \infty, \quad (3)$$

where $T(r) = \sup_{n>0} (r^n / A_n)$.

Theorem 3. In order that the class $D_\alpha\{A_n\}$ ($0 \leq \alpha < \infty$) be quasi-analytic at the point 0, it is necessary and sufficient that the class $D_{\alpha+2}\{A_n\}$ be trivial, i.e. (by virtue of Theorem 2) that

$$\int_0^\infty r^{-(\alpha+2)/(\alpha+1)} \log T(r) dr = \infty. \quad (4)$$

Remarks.

- 1) As is known ((1), p. 55), condition (3) is equivalent to the inequality

$$\sum_{n=1}^{\infty} \beta_n^{-1/(\alpha-1)} < \infty,$$

and condition (4) is equivalent to the equality

$$\sum_{n=1}^{\infty} \beta_n^{-1/(\alpha+1)} = \infty, \quad \text{where } \beta_n = \inf_{k \geq n} A_k^{1/k}.$$

- 2) For $\alpha > 1$ condition (2) remains necessary for the nontriviality of the class $D_\alpha\{A_n\}$, but ceases to be sufficient.
- 3) For $\alpha = 0$ Theorem 3 gives the known classical condition of quasi-analyticity on a line. For $\alpha = 1$ we obtain the condition of quasi-analyticity in a half-plane, established earlier (2)*.
- 4) For $\alpha = 1$ and $\alpha = 0$ Theorem 1 is essentially known, since it follows easily from the more precise results of A. N. Kolmogorov (3) and A. Gorny (4), who established inequalities for upper bounds of successive derivatives of a function on an axis and a half-axis.

3°. We proceed to the proof of Theorem 1. Let $0 < \alpha \leq 1$ and

$$\inf_{n>0} (n^{1-\alpha} A_n^{1/n}) = 0. \quad (5)$$

We shall show that the class $D_\alpha\{A_n\}$ is trivial. Obviously, we may assume that the infimum in (5) is not attained, i.e. that $A_n > 0$, ($n > 0$).

Lemma. Under condition (5) there exists a function $F(z)$, analytic inside the angle $K \setminus I_\alpha$ (the complement of I_α to K) and continuous on its boundary, as well as a sequence of positive numbers $\{B_n\}_0^\infty$, such that

$$F(0) = 1, \quad |F(z)| \leq B_n |z|^{-n} \quad (z \in K \setminus I_\alpha; n = 0, 1, \dots), \quad (6)$$

$$\inf_{n>0} \left[\frac{1}{n} (A_n B_n)^{1/n} \right] = 0. \quad (7)$$

Proof of the lemma. From the classical Watson problem it follows ((1), pp. 55–56) that, for the possibility of constructing a function $F(z)$ satisfying in the angle $K \setminus I_\alpha$ the inequalities (6), it is sufficient that the numbers B_n satisfy the conditions

$$B_1 \leq B_2^{1/2} \leq B_3^{1/3} \leq \dots; \quad \sum_{n=1}^{\infty} B_n^{-1/n(2-\alpha)} < \infty.$$

We shall show that it is possible to satisfy simultaneously these conditions and condition (7). By virtue of (5), one can indicate a sequence of indices $n_0 = 0 < n_1 < n_2 < \dots$ such that $A_{n_k} \leq [k^3(2-\alpha)n_k^{1-\alpha}]^{-n_k}$ ($k = 1, 2, \dots$). Put $B_n = (k^2 n_k)^{n(2-\alpha)}$ ($n_{k-1} < n \leq n_k$; $k = 1, 2, \dots$).

* In paper (2) the problem of quasi-analyticity for a disk is solved; it is not difficult, however, to show that for a half-plane the conditions of quasi-analyticity are the same as for a disk.

Obviously,

$$\sum_{n=1}^{\infty} B_n^{-1/n(2-\alpha)} = \sum_{k=1}^{\infty} \sum_{n=n_{k-1}+1}^{n_k} \frac{1}{k^2 n_k} < \sum_{k=1}^{\infty} \frac{1}{k^2} < \infty.$$

On the other hand,

$$\frac{1}{n_k} (A_{n_k} B_{n_k})^{1/n_k} \leq k^{-(2-\alpha)} \rightarrow 0.$$

The lemma is proved.

Now let $f(z) \in D_{\alpha}\{A_n\}$ ($0 < \alpha \leq 1$). Consider the integral

$$\int_{\gamma_{\alpha}} \frac{F(\varepsilon\zeta)f(\zeta) d\zeta}{(\zeta - z)^2} \quad (\varepsilon > 0),$$

where z is an interior point of I_{α} , and the boundary γ_{α} is traversed in the positive direction. Since

$$f(\zeta) = f(0) + f'(0)\zeta + \dots + \frac{f^{(n-1)}(0)}{(n-1)!} \zeta^{n-1} + R_n(\zeta),$$

$$|R_n(\zeta)| \leq \frac{CA_n}{n!} |\zeta|^n \quad (\zeta \in I_{\alpha}),$$

it follows that

$$\left| \int_{\gamma_{\alpha}} \frac{F(\varepsilon\zeta)f(\zeta) d\zeta}{(\zeta - z)^2} \right| = \left| \int_{\gamma_{\alpha}} \frac{F(\varepsilon\zeta)R_n(\zeta) d\zeta}{(\zeta - z)^2} \right| \leq \frac{CA_n B^n}{\varepsilon^n n!} \int_{\gamma_{\alpha}} \frac{|d\zeta|}{|\zeta - z|^2},$$

because integrals of the form

$$\int_{\gamma_{\alpha}} \frac{F(\varepsilon\zeta)p(\zeta) d\zeta}{(\zeta - z)^2},$$

where $p(\zeta)$ is an arbitrary polynomial, are equal to zero. Taking the infimum over n and using (7), we obtain

$$\left| \int_{\gamma_{\alpha}} \frac{F(\varepsilon\zeta)f(\zeta) d\zeta}{(\zeta - z)^2} \right| \leq C \int_{\gamma_{\alpha}} \frac{|d\zeta|}{|\zeta - z|^2} \inf_{n>0} \left(\frac{1}{\varepsilon} \sqrt[n]{\frac{A_n B^n}{n!}} \right)^n = 0.$$

Passing to the limit as $\varepsilon \rightarrow 0$, we find

$$\int_{\gamma_\alpha} \frac{f(\zeta) d\zeta}{(\zeta - z)^2} = 0 \quad \text{or} \quad f'(z) = 0 \quad (z \in I_\alpha).$$

With a slight modification of this argument, one can also cover the case $\alpha = 0$.

Thus, the necessity of condition (3) is proved. Its sufficiency follows from the fact that for $0 \leq \alpha \leq 1$ there exist functions $g(z)$ such that

$$\sup_{z \in I_\alpha} |g^{(n)}(z)| \leq Ca^n / n^{n(1-\alpha)} \quad (n = 0, 1, 2, \dots), \quad (8)$$

where C, a are certain constants. Obviously, $g(\varepsilon z)$, for sufficiently small $\varepsilon > 0$, will belong to the class $D_\alpha\{A_n\}$, if inequality (2) is satisfied. An example of such a function may be the Mittag-Leffler function ([5], p. 265)

$$E_{2-\alpha}(-z) = \sum_{n=0}^{\infty} \frac{(-1)^n z^n}{\Gamma[1 + (2-\alpha)n]}.$$

This is an entire function of order $(2-\alpha)^{-1}$, bounded in the angle I_α . In an additional angle $K \setminus I_\alpha$ it satisfies the estimate

$$\left| E_{2-\alpha}(-z) - \frac{1}{2-\alpha} e^{(-z)^{1/(2-\alpha)}} \right| < \frac{M}{|z|}.$$

Using these properties, it is easy to prove, by applying Cauchy's integral over a circle of suitable radius to estimate derivatives, that $E_{2-\alpha}(-z)$ satisfies inequalities (8).

The second assertion of Theorem 1 is easily reduced to the first. For this purpose one considers the Cauchy-type integral

$$\int_{\gamma_\alpha} \frac{f(\zeta) d\zeta}{(\zeta - z)^2}$$

separately in the domain $I_\alpha - \gamma_\alpha$ and in the domain $K \setminus I_\alpha$.

4°. To prove Theorems 2 and 3, suppose that $f(z) \in D_\alpha\{A_n\}$, $0 \leq \alpha < \infty$, and consider the Laplace transform of the function $f(z) = g(re^{i\varphi})$, (l_φ is the ray $\varphi = \text{const}$, $0 \leq r < \infty$):

$$F(\zeta) = F(\rho e^{i\theta}) = \int_{l_\varphi} f(z) e^{-z\zeta} dz = e^{i\varphi} \int_0^\infty f(re^{i\varphi}) e^{-r\rho e^{i(\varphi+\theta)}} dr. \quad (9)$$

Obviously, for each fixed φ ($-\alpha\pi/2 \leq \varphi \leq \alpha\pi/2$), this integral converges in the angle $-\pi/2 - \varphi < \theta < \pi/2 - \varphi$, $0 < \rho < \infty$, of the Riemann surface of the function $\log \zeta$. Integrating (9) by parts, we obtain

$$|F(\zeta)| \leq \frac{CA_0}{\rho \cos(\varphi + \theta)}; \quad \left| F(\zeta) - \frac{f(0)}{\zeta} \right| \leq \frac{CA_1}{\rho^2 \cos(\varphi + \theta)}; \dots$$

$$\dots; \quad \left| F(\zeta) - \frac{f(0)}{\zeta} - \dots - \frac{f^{(n-1)}(0)}{\zeta^n} \right| \leq \frac{CA_n}{\rho^{n+1} \cos(\varphi + \theta)}.$$

Denoting by P_φ the half-plane on the Riemann surface of the function $\log \zeta$, defined by the inequalities $-\pi/2 - \varphi < \theta < \pi/2 - \varphi$, $\rho \cos(\varphi + \theta) \geq 1$, we obtain

$$|F(\zeta)| \leq CA_0; \quad \left| F(\zeta) - \frac{f(0)}{\zeta} \right| \leq \frac{CA_1}{\rho}; \dots$$

$$\dots; \quad \left| F(\zeta) - \frac{f(0)}{\zeta} - \dots - \frac{f^{(n-1)}(0)}{\zeta^n} \right| \leq \frac{CA_n}{\rho^n} \quad (\zeta \in P_\varphi). \quad (10)$$

Rotating the line l_φ in (9), we find, by virtue of the analyticity of $f(z)$, that $F(\zeta)$ can be analytically continued to the angle $-(\alpha + 1)\pi/2 < \theta < (\alpha + 1)\pi/2$, and in the domain

$$S_\alpha = \bigcup_{|\varphi| \leq \alpha\pi/2} P_\varphi$$

the estimates (10) are preserved.

It is now easy to prove the necessity in Theorem 2 and the sufficiency in Theorem 3. Indeed, if $f(0) = f'(0) = \dots = 0$, then $F(\zeta)$, by (10), satisfies the conditions of Watson's problem in the domain S_α . It is easy to show that the conditions for the solvability of Watson's problem for the domain S_α and for the angle $I_{\alpha+1}$ are the same; consequently, when condition (4) is fulfilled, one must have $F(\zeta) \equiv 0$, $f(z) \equiv 0$.

Under the conditions of Theorem 2 one must consider the function

$$F_1(\zeta) = F_1(\rho e^{i\theta}) = F(\rho e^{i\theta}) - F(\rho e^{i(\theta+2\pi)})$$

($\theta + 2\pi$ denotes passage to the next sheet of the Riemann surface). Denoting by \tilde{S}_α the domain consisting of the points $\zeta = \rho e^{i\theta}$ for which $\rho e^{i\theta} \in S_\alpha$, $\rho e^{i(\theta+2\pi)} \in S_\alpha$, we obtain, by virtue of (10),

$$|F_1(\zeta)| \leq 2CA_n/|\zeta|^n \quad (n = 0, 1, 2, \dots; \zeta \in \tilde{S}_\alpha).$$

Using Watson's problem once again, we obtain (3).

The sufficiency in Theorem 2 follows from the fact that, when inequality (3) is fulfilled, one can construct in the angle $-(\alpha - 1)\pi/2 \leq \theta \leq (\alpha - 1)\pi/2$ an analytic function $F(\zeta) = F(\rho e^{i\theta})$ satisfying the conditions

$$\int_0^\infty |F(\rho e^{i\theta})| \rho^n d\rho \leq A_n \quad (|\theta| \leq (\alpha - 1)\pi/2; n = 0, 1, 2, \dots).$$

Then the Laplace transform

$$f(z) = f(re^{i\varphi}) = e^{i\theta} \int_0^\infty F(\rho e^{i\theta}) e^{-r\rho e^{i(\theta+\varphi)}} d\rho$$

will satisfy the estimates (1) in a certain angle of opening $\alpha\pi$. If $\alpha \geq 2$, then the function

$$f_1(re^{i\varphi}) = f(re^{i\varphi}) - f(re^{i(\varphi+2\pi)})$$

will satisfy the estimates (1) in an angle of opening $(\alpha - 2)\pi$, but for it, in addition,

$$f_1(0) = f_1'(0) = f_1''(0) = \dots = 0.$$

This completes the proof of Theorems 2 and 3.

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Note: Figure translations are in progress. See original paper for figures.

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