ON THE DIRECT AND INVERSE PROBLEM OF V. A. MARKOV IN THE COMPLEX DOMAIN

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Abstract

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MATHEMATICS

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ON THE DIRECT AND INVERSE PROBLEM OF V. A. MARKOV IN THE COMPLEX DOMAIN

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1. Let continuous complex-valued functions $f(z)$, $\{\varphi_j(z)\}_0^n$ be given on a bicompact Hausdorff space G ([1]). We regard the functions $\{\varphi_j(z)\}_0^n$ as linearly independent. Consider the following generalization of the work of V. A. Markov ([2]). Among all polynomials

$$F(a;z) = \sum_{j=0}^{n} a_j \varphi_j(z), \quad z \in G,$$

whose coefficients satisfy the linearly independent relations

$$\sum_{j=0}^{n} a_j \alpha_j^{(t)} = \alpha_t, \quad t = 1, \ldots, p, \tag{1}$$

find a polynomial $F(a^*;z)$ for which the condition

$$\min_{a_j} \max_{z\in G} \left| \sum_{j=0}^{n} a_j \varphi_j(z) - f(z) \right| = \max_{z\in G} \left| \sum_{j=0}^{n} a_j^* \varphi_j(z) - f(z) \right| = \rho \tag{2}$$

is fulfilled.

Polynomials $F(a;z)$ whose coefficients satisfy the relations (1) will be called **admissible**.

Works ([2–16]) and others are devoted to the problem of Chebyshev approximation without constraints and with constraints.

In the present paper* the results of the author' s work ([11,12]) are generalized to problem (2) with constraints (1). The following notation is used:

$$\operatorname{Re}\varphi_j(z)=\varphi_j'(z),\quad \operatorname{Im}\varphi_j(z)=\varphi_j''(z),\quad \varphi_j(z)=\varphi_j'(z)+i\varphi_j''(z),$$

$$\alpha_j^{(t)}=\alpha_j^{(t)'}+i\alpha_j^{(t)''},\quad \alpha_t=\alpha_t'+i\alpha_t'',\quad \delta(z_s)=F(a;z_s)-f(z_s),$$

$$\operatorname{sgn}\delta(z_s)=e^{i\theta_s},\quad \theta_s=\arg\delta(z_s),\quad \varphi_j^*(z)=\varphi_j(z)\operatorname{sgn}\overline{\delta(z)},$$

$$\operatorname{sgn}\overline{\delta(z)}=e^{-i\theta_s},\quad \varphi_j^*(z)=\varphi_j^{*'}(z)+i\varphi_j^{*''}(z),\quad F^*(a;z)=F(a;z)\operatorname{sgn}\overline{\delta(z)}.$$

Consider the systems of equations:

$$\sum_{s=1}^{r}K_s^{(t)}\varphi_j^{*'}(z_s)+\sum_{s=r+1}^{2n+2}K_s^{(t)}d_j'(s)=\alpha_j^{(t)'},\quad j=0,\dots,n; \tag{3}$$

$$\sum_{s=1}^{r}K_s^{(t)}\varphi_j^{*''}(z_s)+\sum_{s=r+1}^{2n+2}K_s^{(t)}d_j''(s)=\alpha_j^{(t)''},\quad j=0,\dots,n;\ t=1,\dots,p.$$

$$\sum_{s=1}^{r}K_s^{(t)*}\varphi_j^{*}(z_s)+\sum_{s=r+1}^{2n+2}K_s^{(t)*}d_j(s)=-\alpha_j^{(t)''},\quad j=0,\dots,n; \tag{4}$$

* The results of the work were reported at a meeting of the Academic Council of the Institute of Mathematics of the Academy of Sciences of the Ukrainian SSR on 23 I 1963.

$$\sum_{s=1}^{r}K_s^{(t)*}\varphi_j^{*}{}'(z_s)+\sum_{s=r+1}^{2n+2}K_s^{(t)*}d_j''(s)=\alpha_j^{(t)}{}',\qquad j=0,\dots,n;\quad t=1,\dots,p.$$

In the systems (3), (4) the number $r\geq 1$, and the $2n+2$ vectors

$$\vec{\varphi}_s^*=[\varphi_0^*{}'(z_s),\varphi_0^*{}''(z_s),\dots,\varphi_n^*{}'(z_s),\varphi_n^*{}''(z_s)],\qquad s=1,\dots,r;$$

$$\mathbf{d}_s=[d_0'(s),d_0''(s),\dots,d_n'(s),d_n''(s)],\qquad s=r+1,\dots,2n+2,$$

are linearly independent, where $d_\nu'(s),d_\nu''(s)$ are real numbers.

Lemma 1. The vectors $\mathbf{K}^{(t)} = [K_1^{(t)}, \dots, K_{2n+2}^{(t)}]$, $\mathbf{K}^{(t)*} = [K_1^{(t)*}, \dots, K_{2n+2}^{(t)*}]$, $t = 1, \dots, p$, are linearly independent.

From (3), (4) we obtain

$$\sum_{s=1}^{r} K_s^{(t)} \varphi_j^*(z_s) + \sum_{s=r+1}^{2n+2} K_s^{(t)} d_j(s) = \alpha_j^{(t)}, \qquad j = 0, \dots, n; \quad t = 1, \dots, p; \tag{5}$$

$$\sum_{s=1}^{r} K_s^{(t)*} \varphi_j^*(z_s) + \sum_{s=r+1}^{2n+2} K_s^{(t)*} d_j(s) = i\alpha_j^{(t)}, \qquad j = 0, \dots, n; \quad t = 1, \dots, p. \tag{6}$$

From (5), (6), analogously to what was done in ([11], [12]), we obtain

$$\sum_{s=1}^{r} K_s^{(t)} \widetilde{F}^{*\prime}(\widetilde{A}; z_s) + \sum_{s=r+1}^{2n+2} K_s^{(t)} \widetilde{\Phi}'(\widetilde{A}; s) \equiv 0, \qquad t = 1, \dots, p; \tag{7}$$

$$\sum_{s=1}^{r} K_s^{(t)*} \widetilde{F}^{*\prime}(\widetilde{A}; z_s) + \sum_{s=r+1}^{2n+2} K_s^{(t)*} \widetilde{\Phi}'(\widetilde{A}; s) \equiv 0, \qquad t = 1, \dots, p, \tag{8}$$

where $\widetilde{F}^{*\prime}(\widetilde{A}; z_s)$, $\widetilde{\Phi}'(\widetilde{A}; s)$ are real forms with free parameters.

Theorem 1. In order that the vectors $\mathbf{K}_s = [K_s^{(1)}, \dots, K_s^{(p)}, K_s^{(1)*}, \dots, K_s^{(p)*}]$, $s = 1, \dots, 2p$, corresponding to the identities (7), (8), be linearly independent, it is necessary and sufficient that the forms $\widetilde{F}^{*\prime}(\widetilde{A}; z_l)$, $\widetilde{\Phi}'(\widetilde{A}; m)$ $(l, m \neq s_1, \dots, s_{2p})$ be linearly independent.

Let $F(a^*; z)$ be a solution of problem (2). The identity is known ([6], [7])

$$\sum_{s=1}^{r} \lambda_s \widetilde{F}^*(\widetilde{A}; z_s) \equiv 0, \qquad \lambda_s > 0, \qquad s = 1, \dots, r, \tag{9}$$

to which correspond the two identities

$$\sum_{s=1}^{r} \lambda_s \widetilde{F}^{*\prime}(\widetilde{A}; z_s) \equiv 0, \qquad \lambda_s > 0, \tag{10}$$

$$\sum_{s=1}^{r} \lambda_s \widetilde{F}^{*\prime\prime}(\widetilde{A}; z_s) \equiv 0, \qquad \lambda_s > 0, \tag{11}$$

with linear dependence in the narrow sense ([7]) among the real forms, where $\{z_s\}_1^r$ are the Chebyshev deviation points* ([7]).

* For brevity, deviation points are the points of maximum deviation $|F(a;z) - f(z)|$ on G.

To the identities (10), (11) there corresponds the matrix

$$\left\| \begin{matrix} \varphi_j^{*'}(z_s) \\ \varphi_j^{*''}(z_s) \end{matrix} \right\|_{j=0,1,\dots,n;\ s=1,2,\dots,r}. \tag{12}$$

whose rank is not less than $r-1$ by virtue of the linear dependence in the narrow sense of the forms $\widehat{F^*}'(A;z_s)$ (or $\widehat{F^*}''(A;z_s)$).

Lemma 2. If the rank of the matrix (12) is equal to $r-1$, i.e.

$$\sum_{s=1}^{r} c_s \vec{\varphi}_s^* = 0,$$

$c_s \neq 0,\ s = 1,\dots,r$, then $c_s = \lambda_s k,\ k = \text{const}$.

Theorem 2. In order that an admissible polynomial $F(a^*;z)$ be least deviating from the function $f(z)$ on G, it is necessary and sufficient that, for some subsystem of deviation points $\{z_s\}_1^r$, one of the following two conditions be fulfilled:

1) in the case of linear independence of the vectors $\vec{\varphi}_s^*,\ s = 1,\dots,r$, corresponding to the chosen deviation points, for the numbers $K_s^{(t)}, K_s^{(t)*}$ from (3), (4) the following conditions must be satisfied:

a) the rank m of the matrix

$$\left\| \begin{matrix} K_s^{(t)} \\ K_s^{(t)*} \end{matrix} \right\|_{t=1,2,\dots,p;\ s=r+1,\dots,2n+2}, \tag{13}$$

corresponding to the added vectors d_s, is less than or equal to $2p-1$;

b) the nonzero numbers

$$\left| \begin{matrix} K_{l_1}^{(t)} & \cdots & K_{l_{2p-1}}^{(t)} & K_s^{(t)} \\ K_{l_1}^{(t)*} & \cdots & K_{l_{2p-1}}^{(t)*} & K_s^{(t)*} \end{matrix} \right|_{t=1,\dots,p;\ s=1,\dots,2n+2} \tag{14}$$

must be of one sign, where in (14) the first $2p-1$ columns are linearly independent and include m linearly independent columns of the matrix (13);

2) there exists a linear dependence in the narrow sense of the vectors $\vec{\varphi}_s^*$

$$\sum_{s=1}^{r} c_s \vec{\varphi}_s^* = 0, \tag{*}$$

where the numbers $c_s,\ s = 1, \dots, r$, are of one sign.

From 1) of Theorem 2 there follows the identity

$$\sum_{s=1}^{r} \lambda_s F^*(a; z_s) \equiv \sum_{j=0}^{n} \gamma_j a_j, \qquad \lambda_s > 0, \qquad \gamma_j = \sum_{t=1}^{p} \mu_t \alpha_j^{(t)}, \qquad j = 0, \dots, n; \tag{15}$$

$$\lambda_s = \sum_{t=1}^{p} \left(\mu_t' K_s^{(t)} + \mu_t'' K_s^{(t)*} \right), \qquad s = 1, \dots, r,$$

and the numbers $\mu_t = \mu_t' + i\mu_t''$ are computed, in the notation of Theorem 2, from the system of equations

$$\sum_{t=1}^{p} \left(\mu_t' K_\nu^{(t)} + \mu_t'' K_\nu^{(t)*} \right) = 0, \qquad \nu = 1, \dots, 2p - 1.$$

From (*) we obtain

$$\sum_{s=1}^{r} c_s F^*(a; z_s) \equiv 0, \qquad c_s > 0. \tag{16}$$

A number of other theorems have been obtained, formulated with and without added vectors d_s, of the same type as in ([11,12]).

For the value of the best approximation ρ in case 1) of Theorem 2, from (15) we obtain

$$\rho = \left[\sum_{t=1}^{p} \mu_t \alpha_t - \sum_{s=1}^{r} \lambda_s \operatorname{sgn} \overline{\delta(z_s)}\, f(z_s) \right] \Big/ \sum_{s=1}^{r} \lambda_s \tag{17}$$

and in case 2) of Theorem 2, from (16) we obtain:

$$\rho = \left[- \sum_{s=1}^{r} c_s \operatorname{sgn} \overline{\delta(z_s)}\, f(z_s) \right] \Big/ \sum_{s=1}^{r} c_s. \tag{18}$$

Corollary. If $f(z) \equiv 0$ and in (1) $\sum |a_t| > 0$, then in Theorem 2 only case 1) occurs.

Remark 1. If $f(z)$, $\varphi_j(z)$, $j = 0, \dots, n$, are only bounded and the set G is of arbitrary nature, then the study of the problem reduces to the preceding one if one considers all distinct vectors

$$\vec{\varphi}^{(\nu)}(z_s) = [\varphi_0^{(\nu)}(z_s), \dots, \varphi_n^{(\nu)}(z_s), f^{(\nu)}(z_s)], \quad z_s \in G,$$

and takes their closure.

Remark 2. On the basis of the results obtained, a generalization to problem (2) has been obtained of the results of work ([8]) on the connection between Chebyshev approximations and the extremal moment problem.

2. Theorem 3. For any function $f(z)$ and polynomial $F(a; z)$ one can indicate p $(1 \le p \le n+1)$ linearly independent relations under which the polynomial $F(a; z)$ will be least-deviating from $f(z)$ on G.

Methods have been obtained for solving the inverse problem of V. A. Markov, analogous to those applied in the real domain ([12]), etc.

Let us consider one of the methods for solving the inverse problem. For a given function $f(z)$ and polynomial $F(a^*; z)$, $z \in G$, choose p linearly independent relations so that the points of maximal deviation $\{z_s\}_1^r$, for which the vectors φ_s^*, $s = 1, \dots, r$, are linearly independent, form a Chebyshev subsystem of points of deviation, i.e., so that under the chosen relations the polynomial $F(a^*; z)$ is a solution of problem (2).

To solve the problem, take numbers $\lambda_s > 0$, $s = 1, \dots, r$, and put

$$\sum_{s=1}^{r} \lambda_s \varphi_j^*(z_s) = \gamma_j, \quad j = 0, \dots, n.$$

Take numbers μ_t and p linearly independent vectors

$$\vec{\alpha}^{(t)} = [\alpha_0^{(t)}, \dots, \alpha_n^{(t)}], \quad t = 1, \dots, p,$$

satisfying the condition

$$\gamma_j = \sum_{t=1}^{p} \mu_t \alpha_j^{(t)}, \quad j = 0, \dots, n.$$

The sought relations will be

$$\sum_{j=0}^{n} a_j \alpha_j^{(t)} = \alpha_t, \quad t = 1, \dots, p,$$

where

$$\alpha_t = \sum_{j=0}^{n} a_j^* \alpha_j^{(t)}.$$

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Note: Figure translations are in progress. See original paper for figures.