# BANACH SPACES WITH A \(J\)-METRIC.

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## Abstract

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MATHEMATICS

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# BANACH SPACES WITH A $J$-METRIC. $J$-NONNEGATIVE OPERATORS

*(Presented by Academician L. S. Pontryagin on 5 XI 1965)*

**1.** Let a Banach space $\mathfrak{B}$ be the direct sum of two of its subspaces: $\mathfrak{B} = \mathfrak{B}_+ \dot{+} \mathfrak{B}_-$, and let $P_\pm$ be the bounded projectors corresponding to this decomposition $(\mathfrak{B}_\pm = P_\pm \mathfrak{B})$. Introduce in $\mathfrak{B}$ the so-called $J$-metric by putting

$$J(x) = \|P_+ x\|^2 - \|P_- x\|^2 \qquad (x \in \mathfrak{B}).$$

(We note that the squares here could be replaced by any other positive power; cf. ([1,2]).) The function $J(x)$ is, obviously, strongly continuous and, moreover, uniformly (strongly) continuous in every ball $\|x\| \leq C$. Depending on the sign assumed by this function on vectors and sets of vectors from $\mathfrak{B}$, these vectors and sets are naturally classified as $J$-positive ($J$-nonnegative), $J$-negative ($J$-nonpositive), and $J$-neutral.

For the case in which $\mathfrak{B}$ is a Hilbert space and $P_\pm$ are orthoprojectors ($\mathfrak{B} = \mathfrak{B}_+ \oplus \mathfrak{B}_-$), the geometry and the theory of operators in spaces $\mathfrak{B}$ with a $J$-metric were first studied by L. S. Pontryagin, M. G. Krein, and the author under the assumption $\min\{\dim \mathfrak{B}_+, \dim \mathfrak{B}_-\} = \varkappa < \infty$ ([3–8]), and then (already without the restriction $\varkappa < \infty$, partially removed earlier in ([6]), §13) this theory received further development and application in a number of works by Soviet and foreign authors (for a complete bibliography see ([9,10])). The passage to a Banach space, apparently, was first undertaken by F. Bonsall ([1]) ($\varkappa = 1$), then by M. L. Brodskii ([11]) and K. Fan ([2]) ($\varkappa < \infty$), and for $\varkappa = \infty$ by M. G. Krein ([12]), the author ([13]), and K. Fan ([14]).

**2.** A $J$-nonnegative ($J$-nonpositive) subspace is called **maximal** if it is not a proper part of any other $J$-nonnegative ($J$-nonpositive) subspace of $\mathfrak{B}$. From Zorn' s lemma one obtains (cf. ([9]), §3, 10) the so-called

**Maximality principle.** *Every $J$-nonnegative ($J$-nonpositive) subspace is contained in some maximal $J$-nonnegative ($J$-nonpositive) subspace.*

The following facts hold (cf. ([9]), §6):

$1^\circ$. *The projector $P_+$ ($P_-$) maps every $J$-nonnegative ($J$-nonpositive) linear manifold $\mathfrak{L}$ from $\mathfrak{B}$ homeomorphically onto the linear manifold $P_+\mathfrak{L} \subset \mathfrak{B}_+$ ($P_-\mathfrak{L} \subset \mathfrak{B}_-$). In particular,* $\dim \mathfrak{L} \leq \dim \mathfrak{B}_+$ ($\dim \mathfrak{L} \leq \dim \mathfrak{B}_-$).

$2^\circ$. *If, under the conditions of assertion $1^\circ$, $P_+\mathfrak{L} = \mathfrak{B}_+$ ($P_-\mathfrak{L} = \mathfrak{B}_-$), then $\mathfrak{L}$ is a maximal $J$-nonnegative ($J$-nonpositive) subspace.*

Denote by $\mathfrak{T}_+$ ($\mathfrak{T}_-$) the class of all subspaces $\mathfrak{L}$ of $\mathfrak{B}$ that are mapped by the projector $P_+$ ($P_-$) one-to-one onto all of $\mathfrak{B}_+$ ($\mathfrak{B}_-$). With the aid of the Hahn-Banach theorem the following assertion is proved:

$3^\circ$. *Every one-dimensional $J$-nonnegative ($J$-nonpositive) subspace is contained in some maximal $J$-nonnegative ($J$-nonpositive) subspace $\mathfrak{L} \in \mathfrak{T}_+$ ($\mathfrak{L} \in \mathfrak{T}_-$).*

For subspaces of larger dimension this assertion is, generally speaking, no longer true.

**Lemma 1.** *In order that every $J$-nonnegative subspace be contained in some maximal $J$-nonnegative subspace of the class $\mathfrak{T}_+$, it is necessary and sufficient that every bounded linear operator acting from an arbitrary subspace $\mathfrak{B}_+^0 \subset \mathfrak{B}_+$ into the subspace $\mathfrak{B}_-$ admit an extension, with the same norm, to an operator acting from all of $\mathfrak{B}_+$ into $\mathfrak{B}_-$.*

**Remark.** As is known ([15,16]), the condition of Lemma 1 will be satisfied if, for example, any one of the following requirements is satisfied:
$\alpha$) $\mathfrak{B}_+$ is a unitary space; $\beta$) $\mathfrak{B}_-$ is a space of type $\mathfrak{M}$.

An assertion analogous to Lemma 1 holds, of course, also for $J$-nonpositive subspaces.

A special role is played by the particular case in which

$$\min\{\dim \mathfrak{B}_+, \dim \mathfrak{B}_-\} = \varkappa < \infty.$$

For definiteness we shall assume $\varkappa = \dim \mathfrak{B}_+$. In the following lemma $0 \leq \varkappa < \infty$.

**Lemma 2.** *If the linear manifold $\mathfrak{L}(\subset \mathfrak{B})$ contains a $\varkappa$-dimensional $J$-positive subspace $*$, then the same property is possessed by every linear manifold $\mathfrak{D}(\subset \mathfrak{L})$ dense in $\mathfrak{L}$ (i.e. such that $\overline{\mathfrak{D}} \supset \mathfrak{L}$).*

3. A linear operator $V$, defined on some linear manifold $\mathfrak{D}_V$ in $\mathfrak{B} = \mathfrak{B}_+ \dot{+} \mathfrak{B}_-$, will be called $J$-nonnegative if from $x \in \mathfrak{D}_V$ and $J(x) \geq 0$ it always follows that $J(Vx) \geq 0$. $J$-nonpositive operators are defined analogously; for them in what follows all assertions are easily obtained from those given in the text.

**Theorem 1.** *Let $V$ be a $J$-nonnegative operator defined on some linear manifold $\mathfrak{D}_V$ in $\mathfrak{B}$, and let $\mathfrak{D}_V \cap \mathfrak{B}_+ \neq \{0\}$. The operator $V$ is bounded if and only if the operator $P_+V$ is bounded.*

From this theorem and Lemma 2 one can obtain the following generalization and strengthening of one result of M. L. Brodskii (see ([11]), $1^\circ$):

**Theorem 2.** *Let $V$ be a $J$-nonnegative operator in $\mathfrak{B} = \mathfrak{B}_+ \dot{+} \mathfrak{B}_-$, and let* $\dim \mathfrak{B}_+ = \varkappa$ $(1 \leq \varkappa < \infty)$. *If $\mathfrak{D}_V \cap \mathfrak{B}_+ \neq \{0\}$ and $\mathfrak{D}_V$ contains a $\varkappa$-dimensional $J$-positive subspace, while the operator $V$ annihilates no $J$-positive vector, then this operator is bounded.*

In the particular case when $\mathfrak{B}$ is a Hilbert space and $\mathfrak{B}_\pm$ are its mutually orthogonal subspaces, the conditions of Theorem 2 imposed on $\mathfrak{D}_V$ are satisfied, for example, when $\overline{\mathfrak{D}}_V = \mathfrak{B}$. The condition requiring that the operator $V$ not annihilate $J$-positive vectors is essential, since even in a Hilbert space $\mathfrak{B}$, for $\mathfrak{D}_V = \mathfrak{B}$, one can give examples of one-dimensional $J$-nonnegative operators which, when this condition is violated, turn out to be unbounded.

**Theorem 3.** *If, in a Banach space $\mathfrak{B} = \mathfrak{B}_+ \dot{+} \mathfrak{B}_-$, a $J$-nonnegative operator $V$ annihilates some $J$-positive vector, then the linear manifold $\mathfrak{R}_V = V\mathfrak{D}_V$ is $J$-nonnegative (and, consequently, by $1^\circ$, $\dim \mathfrak{R}_V \leq \dim \mathfrak{B}_+$) (cf. ([11]), $1^\circ$).*

**Corollary.** *If, under the conditions of Theorem 3, $\mathfrak{D}_V = \mathfrak{B}$, then the linear manifold $\mathfrak{R}_V$ is a $J$-nonnegative invariant linear manifold for the operator $V$.*

4. The question touched upon in the corollary to Theorem 3, concerning the existence for a $J$-nonnegative operator of a $J$-nonnegative invariant subspace, is connected with a well-known problem in operator theory which has given rise to an extensive literature ([1–8,10–14,17]). The central point in these works is the proof of the existence of maximal $J$-nonnegative invariant subspaces of the class $\mathfrak{T}_+$ for various classes of operators in spaces with a $J$-metric. The most fruitful idea here proved to be that of M. G. Krein ([4–6,12]) [[unclear: continuation cut off at bottom]]

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* A linear manifold (subspace) will be called $J$-positive if all its nonzero vectors are $J$-positive. $J$-negative linear manifolds are defined analogously.

···of obtaining such proofs by means of various fixed-point principles.

The greatest progress in the question under consideration has been achieved in the paper ([12]) (see also ([13]))*, where, in proving the existence of $J$-nonnegative invariant subspaces of class $\mathfrak{J}_+$ for a $J$-nonnegative operator $V$ with $\mathfrak{D}_V = \mathfrak{B}$, the following requirements are imposed: 1) $V$ is bounded; 2) the operator $P_+VP_-$ admits approximation in norm by finite-dimensional operators (in the case when $\mathfrak{B}$ is a Hilbert space or a Banach space with a basis, this is equivalent to the requirement that $P_+VP_-$ be a completely continuous operator).

The assertion contained in a recent paper of K. Fan (([14]), Corollary 2 of Theorem 2), if valid, would constitute, in the case of its applicability, a further step toward the generalization of the results named in ([9,12,13,17]), since condition 2) is absent from the formulation of this assertion of K. Fan. However, unfortunately, the proof of Corollary 2 in ([14]), as its analysis shows, contains a fundamental error.

Thus, even for $J$-unitary operators $V$ $(\mathfrak{D}_V = \mathfrak{R}_V = \mathfrak{B},\ J(Vx) = J(x),\ x \in \mathfrak{B})$ in a Hilbert space $\mathfrak{B} = \mathfrak{B}_+ \oplus \mathfrak{B}_-$, the possibility of dispensing with condition

2) remains problematic (condition 1) is satisfied automatically in this case [(18)]). Closely connected with this question is another unsolved problem, already for families of commuting $J$-unitary operators, posed by R. S. Phillips [(19)] (see the commentary to the translation [(20)] of this article).

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* We take this opportunity to make the following clarification: in Theorem 3 of [(13)], if $\mathfrak{B} = \mathfrak{B}_+ \dot{+} \mathfrak{B}_-$ is a Banach space, then one must additionally require of the operator $A$ that it send every $J$-nonnegative subspace of class $\mathfrak{J}_+$ into a subspace of class $\mathfrak{J}_+$. In a Hilbert space $\mathfrak{B} = \mathfrak{B}_+ \oplus \mathfrak{B}_-$ this requirement is satisfied automatically by virtue of the maximality principle (see item 2) and item a) of the remark to Lemma 1 of the present note.