SOME LIMIT THEOREMS ON THE DISTRIBUTION OF FUNCTIONALS OF PROCESSES ASYMPTOTICALLY CLOSE TO MARKOV PROCESSES

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Abstract

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SOME LIMIT THEOREMS ON THE DISTRIBUTION OF FUNCTIONALS OF PROCESSES ASYMPTOTICALLY CLOSE TO MARKOV PROCESSES

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Let $R(0,T)$ be some complete metric separable space of measurable real functions on $[0,T]$. Below we set out conditions singling out a broad class of processes $\{Z_T(t), 0 \le t \le T\}$ in $R(0,T)$, in a certain sense close, for large T, to Markov processes, for which the distribution of functionals f, continuous in the uniform metric, of the correspondingly normalized processes $Z_T(t)$ converges as $T \to \infty$ to the distribution of the functionals $f(w)$ of a certain classical diffusion process $w(t)$.

In many cases the processes $Z_T(t)$ under consideration are functions of other processes $X_T(t)$, defined on a more complicated phase space X, but having there a simpler probabilistic nature (for example, $Z_T(t)$ may be one of the coordinates of $X_T(t)$). Therefore it is often natural to relate the dependence conditions not to $Z_T(t)$, but directly to the process $X_T(t)$.

Thus, let X be some space and $\mathfrak{B}$ a σ-algebra of subsets given on it. Suppose that for each $t \in [0,T]$ a space $(X^t, \mathfrak{B}^t)$ of functions $x(u)$ is defined in such a way that the following properties hold: for every $u \in [0,t]$ and $B \subset \mathfrak{B}$, $0 \le u \le t$, with values in X, $\{x : x(u) \in B\} \in \mathfrak{B}^t$, $\mathfrak{B}^{t_1} \subset \mathfrak{B}^{t_2}$ for $t_1 < t_2$. If a distribution P is given on $(X^T, \mathfrak{B}^T)$, then thereby a process $\{X_T(t), 0 \le t \le T\}$ is given in the phase space X, which we shall call the **basic** process.

Assume further that there exists a measurable mapping of $(X^T, \mathfrak{B}^T)$ onto the space $(R(0,T), \mathfrak{M}_T^0)$, where $\mathfrak{M}_0^T$ is the σ-algebra generated by the open sets of $R(0,T)$, determined by a family of functionals $F_{t,T}$

$$Z_T(t) = F_{t,T}(X_T(u), 0 \le u \le T), \tag{1}$$

which, moreover, for every t are measurable with respect to $\mathfrak{B}^t$ ($\{x : Z_T(t) < c\} \in \mathfrak{B}^t$). Thus, for fixed t and T, the random variable $Z_T(t)$ may be represented as a function of the trajectory $X_T(u)$ on the interval $[0,t]$. The process $Z_T(t)$

defined by relation (1) will be assumed separable. In X_T, Z_T the index T means that a "scheme of series" is being considered.

Before proceeding to the formulations, let us introduce some concepts. Let $s > 0$, and let $Y^s(t)$, $s \le t \le T$, be the projection of $X_T(u)$, $0 \le u \le T$, onto the space $X^{(s)}$ of functions on $(t-\delta, t]$, so that the value $Y^s(t)$ in the phase space $X^{(s)}$ is the trajectory of the basic process on the interval $(t-s, t]$. We shall call the process $X_T(t)$ $(\mathfrak{D}, \alpha)$-**returning** if the return time of $Y^s(t)$ to a measurable set $\mathfrak{D} \subset X^{(s)}$ has a uniformly bounded moment of order α. More precisely, let $n_{\mathfrak{D}}(u)$ be the time of first hitting of $\mathfrak{D}$ by $Y^s(t)$ after time u:

$$n_{\mathfrak{D}}(u) = \begin{cases} \inf(t \ge u : \{Y^s(t) \in \mathfrak{D}\}), & \text{if this lower bound is } \le T, \\ T, & \text{if } Y^s(t) \notin \mathfrak{D} \text{ for every } t,\ u \le t \le T. \end{cases}$$

We shall assume the set $\mathcal{D}$ to be such that $n_{\mathcal{D}}(u)$, the time of the "second hit" $n_{\mathcal{D}}(n_{\mathcal{D}}(u)+s)$, etc., are random variables*. Let $\mathfrak{M}(u)$ be the σ-algebra generated by the events $\{n_{\mathcal{D}}(u) \ge v; (X_T(t), 0 \le t \le v) \in B\}$, where $u \le v \le T$, $B \in \mathfrak{B}^v$, and let

$$q_{\mathcal{D}}(u) = n_{\mathcal{D}}(n_{\mathcal{D}}(u)+s) - n_{\mathcal{D}}(u) \ge s$$

be the time that has elapsed until the return of $Y^s(t)$ to $\mathcal{D}$ after the instant $n_{\mathcal{D}}(u)$. Then the process $X_T(t)$ is called $(\mathcal{D}, a)$-recurrent if, almost everywhere,

$$\mathbf{M}_{\mathfrak{M}(u)} q^a_{\mathcal{D}}(u) < c.$$

Here and below, by the letter c we denote various constants independent of the variables under consideration and of T. In order to use the property of $(\mathcal{D}, a)$-recurrence, we must be sure that $Y^s(t)$ first enters $\mathcal{D}$ sufficiently quickly. Therefore, $(\mathcal{D}, a)$-recurrence will also be assumed to include the relation $\mathbf{P}(n_{\mathcal{D}}(0) > \Delta T) \to 0$ as $T \to \infty$ for every $\Delta > 0$. The trajectory $Z_T(t)$ between two neighboring hits of $Y^s(t)$ in $\mathcal{D}$ will be called a cycle. Here only hits that are at least s apart from the given one are regarded as neighboring (see the definition of $q_{\mathcal{D}}(u)$).

We proceed to formulate the conditions. Their meaning consists in finding such a set $\mathcal{D}$ (if it exists) for which entering it weakens the dependence on the entire past history, except for the value of the process at the last moment of time. The conditions will be imposed on the increments

$$Z_{u,t} = Z_T(n_{\mathcal{D}}(u)+t) - Z_T(n_{\mathcal{D}}(u)),$$

which make sense only when $n_{\mathcal{D}}(u) + t \le T$. To avoid constant and essentially immaterial qualifications, we shall, without loss of generality, assume that the processes $X_T(t)$ and $Z_T(t)$ are defined on the whole half-line $(0, \infty)$, where their continuation to (T, ∞) is chosen so that the properties I-III below are preserved on the whole domain $(0, \infty)$. It is not hard to see that such a continuation is always possible.

Let $\varepsilon_T(t)$ be a function having the property that $\varepsilon_T(t) \to 0$ as $T \to \infty$, $T^{\theta_1} \leq t \leq T^{\theta_2}$ for some $0 < \theta_1 < \theta_2 < 1$. Suppose there exist $s > 0$, a set $\mathcal{D} \in X^{(s)}$, and $a > 1$ such that the process $X_T(t)$ is $(\mathcal{D}, a)$-recurrent and, for every u,

$$\text{I.} \quad \mathbf{M}_{\mathfrak{M}(u)} Z_{u,t}^2 = B^2(t)\,[\,b(\zeta(u)) + r^1_{\mathfrak{M}(u)}\,],$$

$$\text{II.} \quad \mathbf{M}_{\mathfrak{M}(u)} \frac{Z_{u,t}}{B(T)} = \left(\frac{B(t)}{B(T)}\right)^2 [\,a(\zeta(u)) + r^2_{\mathfrak{M}(u)}\,],$$

where $\zeta(u) = Z_T(n_{\mathcal{D}}(u))/B(T)$; almost everywhere $r^i_{\mathfrak{M}(u)} < \varepsilon_T(t)$, $i = 1, 2$; and $B^2(t)$ is a function representable in the form $B^2(t) = th(t)$, where $h(e^t)$ is a function slowly varying in the sense of Karamata.

The properties of the finite functions a and b occurring in I, II will be specified below.

We shall also assume that the following condition is satisfied:

$$\text{III.} \quad \text{For some } \gamma > 0, \text{ almost everywhere } \mathbf{M}_{\mathfrak{M}(u)} \sup_{0 \leq t \leq q_{\mathcal{D}}(u)} |Z_{u,t}|^{2+\gamma} < c;$$

$$\text{for every } \Delta > 0 \text{ and } T \to \infty, \quad \mathbf{P}\left(\sup_{0 \leq t \leq n_{\mathcal{D}}(0)} |Z_T(t) - Z_T(0)| > \Delta B(T)\right) \to 0.$$

* For this it is sufficient, for example, that there exist a countable set Q, everywhere dense in $[0, T]$, such that for every interval Δ

$$\bigcap_{t_k \in Q \cap \Delta} \{Y^s(t_k) \in \mathcal{D}\} = \bigcap_{t \in \Delta} \{Y^s(t) \in \mathcal{D}\} \in \mathfrak{B}^T.$$

However, this restriction may in general be ignored, since from the very beginning $Y^s(t)$ can be defined, say, only at integer points (then $n_{\mathcal{D}}(u)$ will take only a finite number of values). We have rejected this variant only because it appears less natural and more cumbersome in exposition.

This condition forbids excessively large excursions of the trajectory $Z_T(u)$ on cycles.

Concerning the functions a, b, we shall assume the following. Let ψ be a twice continuously differentiable, bounded function, all of whose change is concentrated on a finite interval, and suppose that there exists a transition density on the real line $p(x,t,y) \geq 0$, continuously differentiable in t and twice continuously differentiable in x for $0 < t \leq 1$, for which, as $t \to \infty$, uniformly in x,

$$\text{IV.} \quad \frac{1}{t}\int (y-x)p(x,t,y)\,dy \to a(x), \qquad \frac{1}{t}\int (y-x)^2 p(x,t,y)\,dy \to b(x) \geq$$

$$\geq b_0 > 0,$$

$$\frac{1}{t}\int (y-x)^{2+\gamma} p(x,t,y)\,dy \to 0,$$

and the second derivative with respect to x of the function $\int p(x,t,y)\psi(y)\,dy$ is uniformly continuous in the domain $0 \leq t \leq 1$, $-\infty < x < \infty$.

It is not difficult to note that the last integral, as well as $\int_y^\infty p(x,t,u)\,du$, together with $p(x,t,y)$, are solutions of the equation

$$\frac{\partial p}{\partial t} = a\frac{\partial p}{\partial x} + \frac{b}{2}\frac{\partial^2 p}{\partial x^2}.$$

Sufficient conditions for the fulfillment of conditions IV are, for example ([1],[2]), boundedness and the Hölder property of the functions $a(x)$ and $b(x)$.

Under the listed conditions the following propositions are valid. Put $Z_T(0) = 0$ and denote by P_T the distribution, induced by the measure P, of the process

$$x_T(t) = Z_T(tT)/B(T), \qquad 0 \leq t \leq 1,$$

in the space $R(0,1)$ with metric $\rho_R(x,y)$. Let, moreover, $C(0,1)$ be the space of continuous functions on $[0,1]$ with the uniform metric

$$\rho_C(x,y) = \sup_t |x(t) - y(t)|,$$

and let W be the distribution of a certain diffusion process $\{w(t),\ 0 \leq t \leq 1\}$ in $C(0,1)$ with the σ-algebra $\mathfrak{A}_0^1$ of cylindrical (or open) sets, constructed from the transition function

$$P(x,t,E) = \int_E p(x,t,y)\,dy.$$

Obviously, $w(t)$ can also be characterized by means of the infinitesimal operator

$$A = \left(a\frac{\partial}{\partial x} + \frac{b}{2}\frac{\partial^2}{\partial x^2}\right),$$

defined on the corresponding set of functions from $C(-\infty, \infty)$.

Theorem 1. Let $R(0,1) \supset C(0,1)$, and suppose that uniform convergence

$$\rho_C(\varphi_n, \varphi) \to 0 \qquad (\varphi_n, \varphi \in R)$$

implies convergence

$$\rho_R(\varphi_n, \varphi) \to 0$$

in the sense of the metric ρ_R, and let f be a ρ_R-continuous functional on $R(0,1)$. Then, when conditions I–IV are satisfied,

$$\mathbf{P}(f(x_T) < y) \Rightarrow \mathbf{P}(f(w) < y).$$

If W can be defined on R by means of finite-dimensional distributions, then the assertion of Theorem 1 will mean weak convergence $P_T \Rightarrow W$ of distributions in R. If $R = D(0,1)$ is the space of functions without discontinuities of the second kind with metric ρ_D, introduced in (3), then the assertion of the theorem is equivalent to the convergence of the so-called I-continuous functionals (introduced by Skorokhod (4) and continuous in the metric ρ_D).

However, functionals continuous in the topology R may form too poor a class.

Theorem 2. Let $R(0,1) \supset C(0,1)$, and let f be a real-valued, $(F(0,1), \mathfrak{A}_1)$-measurable functional defined on $R(0,1)$ and continuous at the "points" of $C(0,1)$ in the sense of the metric ρ_C: $f(\varphi_n) \to f(\varphi)$, if $\rho_C(\varphi_n, \varphi) \to 0$, $\varphi_n \in R(0,1)$, $\varphi \in C(0,1)$. Then, when conditions I–IV are satisfied, the weak convergence (2) holds.

As a consequence of Theorem 2 we give one assertion concerning processes of a somewhat different kind. We shall assume the original process $Z_T(t)$ to be defined in the space $(R(0,2T), \mathfrak{M}_0^{2T})$ and to satisfy there conditions I-IV. Put

$$Z_T^g(t) = \int_0^{2T} Z_T(u) g(t-u)\, du, \qquad 0 \leqslant t \leqslant T,$$

where $g(t)$ is a summable function for which

$$\int_{-\infty}^{\infty} g(t)\, dt = 1, \qquad \int_{-\infty}^{\infty} |t|^{1/2+\varepsilon} g(t)\, dt < \infty$$

for some $\varepsilon > 0$.

Theorem 3. *Suppose the relation*

$$y(t) = \int_0^{2T} x(u) g(t-u)\, du, \qquad 0 \leqslant t \leqslant T,$$

defines a measurable mapping $(R(0,2T),\mathfrak{M}_0^{2T})$ *into* $(C(0,T),\mathfrak{N}_0^T)$. *Then, with respect to the distribution* P_T^g *in* $C(0,1)$ *of the process* $x_T^g(t)=Z_T^g(tT)/B(T)$, *the assertion* $P_T^g\Rightarrow W$ *holds.*

Results of the same type have also been obtained for convergence to diffusion processes with boundary conditions, for example with reflection from a sufficiently smooth curve.

The questions considered in this note arose as a result of the investigation of a class of problems connected with semi-Markov processes and various generalized renewal processes, for which conditions I–III for convergence of distributions seem to me the most convenient.

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