PROOF OF THE UPPER SEMICONTINUITY OF THE VARIATION OF A SET

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Abstract

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PROOF OF THE UPPER SEMICONTINUITY OF THE VARIATION OF A SET

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In this note it is proved that the k-dimensional variation $V_k(e)$ of a closed set e, situated in Euclidean space, is upper semicontinuous if the variations of smaller dimension are bounded. Let us recall the definition of k-dimensional variation (see (^{1})).

Let E_n be n-dimensional Euclidean space; Ω_n^k the space of $(n-k)$-dimensional planes β_{n-k} in E_n; μ_n^k the Haar measure in the space Ω_n^k, invariant with respect to the group of transformations of the space Ω_n^k generated by motions of the space E_n; $V_0(e)$ the number of components of the set e. The k-th variation of a closed set e is the number

$$V_k(e) = \int_{\Omega_n^k} V_0(e \cap \beta_{n-k})\, d\mu_n^k.$$

The normalization of the measure μ_n^k is chosen so that the k-th variation of a k-dimensional unit cube is equal to 1.

In the space of subsets of E_n we shall use the deviation metric, i.e., as the distance between sets e and f from E_n we take the quantity $r(e,f) = \max[\rho(e,f), \rho(f,e)]$, where $\rho(A,B) = \sup_{x \in A} \rho(x,B)$, and $\rho(x,B)$ is the distance from the point x to the set B.

After these explanations we formulate precisely our result.

Theorem 1. If, for closed sets e and $e_1, e_2, \dots, e_i, \dots$,

1) $\lim\limits_{i\to\infty} r(e, e_i) = 0$;
2) all $V_m(e_i)$ $(m = 0, 1, 2, \dots, (k-1);\ i = 1, 2, \dots)$ do not exceed some constant M,

then

$$\sup_{i\to\infty} V_k(e_i) \geq V_k(e).$$

Consider the minimal σ-algebra $\nu_{k,n}$ of sets in E_n containing all closed subsets of E_n with finite variations whose dimension is not greater than k.

The variation $V_k(e)$, extended to all sets $e \in \nu_{k,n}$, turns out to be a countably additive measure of order k, which is semicontinuous (see the formulation of the theorem) and on all polyhedra coincides with k-dimensional Lebesgue measure. It seems plausible that a measure satisfying the listed conditions is unique, i.e., coincides with $V_k(e)$.

Notation.

$$d(g) = \sup_{a \in g,\ b \in g} \rho(a, b)$$

is the diameter of the set g; $\partial(e)$ is the boundary of the set e; the depth of immersion of the set e in $g \cap \beta$ will be called

$$r(e, g, \beta) = \sup \rho(x, \partial(g \cap \beta)),$$

where the least upper bound is taken over all $x \in e \cap g \cap \beta$;

$$d(e, g, \beta_p) = \{\mathrm{mes}_p(g \cap \beta_p) - \mathrm{mes}_p(e \cap g \cap \beta_p)\}^{1/p},$$

where $\beta_p \subset E_n$ is a p-dimensional plane;

$$m(e, g, \beta) = \min\{[\Gamma(e, g, \beta)],\ [d(e, g, \beta)]\},$$

$$k(g, \beta) = \Gamma(g, g, \beta)[d(g)]^{-1}.$$

Lemma 1. Let there be given a convex domain $g \subset E_n$ such that $k(g, E_n) \geq \gamma > 0$, and a set $e \subset E_n$ such that every component of the set $e \cap g$ meets $\partial(g \cap E_n)$.

Denote by $\Omega(\Delta_1, \Delta_2, g)$ the set of hyperplanes β_{n-1} determined by the conditions: $k(g, \beta_{n-1}) \geq \Delta_1$ and $m(e, g, \beta_{n-1}) \geq \Delta_2 m(e, g, E_n)$.

There exist numbers $\varepsilon > 0$, $\Delta_1 > 0$, and $\Delta_2 > 0$, depending only on n and γ, such that

$$\mu_n^1[\Omega(\Delta_1, \Delta_2, g)] \geq \varepsilon m(e, g, E_n).$$

Lemma 2. Let the set $e \subset E_n$ be closed and bounded, and let $g_1, g_2, \dots, g_p$ be convex, pairwise disjoint sets, with

$$k(g_i, E_n) \geq \gamma > 0 \quad (i = 1, 2, \dots, p).$$

Then for every $k = 0, 1, 2, \dots, n-1$,

$$\mathbf{V}_k(e) \geq \sum_{i=1}^{p} C_{i,k}[m(e, g_i, E_n)]^k,$$

where $\{C_{i,k}\}$ are nonnegative numbers such that, for every i,

$$\sum_{k=0}^{n-1} C_{i,k} \geq C(n,\gamma) > 0$$

($C(n,\gamma)$ depends only on n and γ).

Proof. The lemma is proved by induction on n.

Consider the case $n = 1$. Among all intervals $\{g_i\}$ intersecting one and the same component of the set e, at most two intervals can satisfy the condition $m(e, g_i, E_1) > 0$, and therefore

$$\mathbf{V}_0(e) \geq \sum_{i=1}^{p} [m(e, g_i, E_1)]^0 \cdot \frac{1}{2}.$$

(Here it is assumed that if $m(e, g, E_1) = 0$, then $[m(e, g_i, E_1)]^0 = 0$); that is, for $n = 1$ the lemma is proved.

Now consider the general case. Recall that

$$\mathbf{V}_k(e) = C(n,k) \int_{\Omega_n^1} \mathbf{V}_{k-1}(e \cap \beta_{n-1})\, d\mu_n^1,$$

where $C(n,k) > 0$ is a normalizing constant independent of e. By the induction hypothesis, for every plane β_{n-1} we have

$$\mathbf{V}_{k-1}(e \cap \beta_{n-1}) \geq \sum_i C_{i,k-1}(\beta_{n-1})[m(e, g_i, \beta_{n-1})]^{k-1},$$

$$\sum_{q=1}^{n-2} C_{i,q}(\beta_{n-1}) \geq C((n-1),\gamma) \quad (i = 1, 2, \dots, p).$$

Here we assume that if $g_i \cap \beta_{n-1}$ is empty, then $m(e, g_i, \beta_{n-1}) = 0$.

Integrating the last inequality, we obtain

$$\frac{\mathbf{V}_k(e)}{C(n,k)} \int_{\Omega_n^1} \mathbf{V}_{k-1}(e \cap \beta_{n-1})\, d\mu_n^1 \geq \sum_i \int_{\Omega_n^1} C_{i,k-1}(\beta_{n-1})[m(e, g_i, \beta_{n-1})]^{k-1}\, d\mu_n^1.$$

Put

$$C_{i,k} = C(n,k)[m(e,g_i,E_n)]^{-k} \int_{\Omega_n^1} C_{i,k-1}(\beta_{n-1})[m(e,g_i,\beta_{n-1})]^{k-1}\, d\mu_n^1$$

$(k = 1, 2, \dots, n-1)$; $\qquad C_{i,0} = 1$, if $g_i \cap e$ contains a component meeting $\partial(g_i, E_n)$ components, and $C_{i,0} = 0$ otherwise; we have

$$V_0(e) \geq \sum_i C_{i,0} = \sum_i C_{i,0}\, [m(e,g_i,E_n)]^0.$$

Thus, for every $k = 0, 1, \dots, n-1$,

$$V_k(e) \geq \sum_i C_{i,k}\, [m(e,g_i,E_n)]^{k-1},$$

and, by virtue of Lemma 1,

$$\begin{aligned}\sum_{k=0}^{n-1} C_{i,k} = C_{i,0} + \sum_{k=1}^{n-1} C(n,k) \int_{\Omega(\Delta_1,\Delta_2,g_i)} C_{i,k-1}(\beta_{n-1})[m(e,g_i,\beta_{n-1})]^{k-1} \times \\ \times [m(e,g_i,E_n)]^{-k}\, d\mu_n^1 \geq C_{i,0} + \sum_{k=1}^{n-1} C(n,k)[m(e_i g_i, E_n)]^{-k} \times \\ \times \int_{\Omega(\Delta_1,\Delta_g,g_i)} C_{i,k-1}(\beta_{n-1})[\Delta_2 m(e,g_i,E_n)]^{k-1}\, d\mu_n^1 \geq \\ \geq C_{i,0} + \sum_{k=1}^{n-1} C(n,k)\Delta_2^{k-1}[m(e,g_i,E_n)]^{-1} \int_{\Omega(\Delta_1,\Delta_2,g_i)} C_{i,k}(\beta_{n-1})\, d\mu_n^1 \geq \\ \geq C_{i,0} + \min_k C(n,k)\Delta_2^{k-1}[m(e,g_i,E_n)]^{-1} \int_{\Omega(\Delta_1,\Delta_2,g_i)} \sum_{k=1}^{n-k} C_{i,k}(\beta_{n-1})\, d\mu_n^1 \geq \\ \geq C_{i,0} + C'(n,\Delta_1)[m(e,g_i,E_n)]^{-1}\varepsilon \cdot m(e,g_i,E_n) C((n-1),\Delta_1,\gamma) \geq C(n,\gamma).\end{aligned}$$

The lemma is proved.

Lemma 3. Let the closed set e be an ε-net for the measurable set $f \subset E_n$. Then, for every $\varepsilon \leq \varepsilon(f)$, for some $k = k(\varepsilon)$ the inequality

$$V_k(e) \geq C(n)[\mathrm{mes}_n\, f - \mathrm{mes}_n\, e]\varepsilon^{k-n},$$

holds, where $C(n)$ depends only on n.

Proof. We shall assume that $\Delta = \operatorname{mes}_n f - \operatorname{mes}_n e > 0$. Fix p equal pairwise nonintersecting cubes $g_1, g_2, \dots, g_p$ with side 4ε and such that

$$\sum_i \operatorname{mes}_n(f \cap g_i) \geq (1 - \tfrac{1}{8}\Delta) \operatorname{mes}_n f.$$

Since

$$\sum_i [m(e, g_i, E_n)]^n \geq C_1(n)\Delta,$$

then by virtue of Lemma 2 one can indicate indices $i_1, i_2, \dots, i_q$ and a number k such that

$$V_k(e) \geq \frac{C(n,\gamma)}{n} \sum_{s=1}^{q} [m(e, g_{i_s}, E_n)]^k,$$

$$\sum_{s=1}^{q} m(e, g_{i_s}, E_n)^n \geq C_2(n)\Delta,$$

and consequently,

$$V_k(e) \geq C(n) \cdot \Delta \cdot \varepsilon^{k-n}.$$

The lemma is proved.

Notation.
Let φ_n^k be the bundle of k-dimensional planes τ_k passing through one and the same point; let $\overline{m}_n^k$ be the measure in the space φ_n^k, invariant with respect to the group of transformations generated by rotations of space; $\beta_{s+n-k}(\beta_s, \tau_k)$ be the plane of dimension $s + n - k$ containing the s-dimensional plane $\beta_s \subseteq \tau_k$ and the $(n-k)$-dimensional plane $\beta_{n-k} \subset E_n$, which is ···

orthogonal complement to τ_k

$$V_{s,\tau_k}(e) = \int_{\beta_s \subset \tau_k} V_0(e \cap \beta_{s+n-k}(\beta_s, \tau_k))\, d\mu_k^{k-s}.$$

Lemma 4. Let the set $f \subset E_n$ be closed and such that, for every plane $\beta_s \subset \tau_k$ (τ_k fixed) and for every pair of components f_1 and f_2 of the set $f \cap \beta_{s+n-k}(\beta_s, \tau_k)$,

$$\min_{\substack{a \in f_1 \\ b \in f_2}} \rho(a,b) \geq \Delta > 0$$

(Δ does not depend on β_s, f_1, f_2), and let the set e be an ε-net for f and lie in the $(\frac{1}{4}\Delta)$-neighborhood of the set f. Then, if $\varepsilon < \varepsilon(f)$ ($\varepsilon(f)$ does not depend on e), then

$$\sum_{s=0}^{k-1} V_{s,\tau_s}(e) \geq C(k)\,[V_{k,\tau_k}(f) - V_{k,\tau_k}(e)]\,\varepsilon^{-1}$$

($C(k)$—see Lemma 3).

Lemma 5. If a closed set is equal to $\lim_{i\to\infty} e_i$, then for every plane τ_k

$$\lim_{i\to\infty} \sum_{s=0}^{k-1} V_{s,\tau_k}(e_i) = +\infty,$$

provided only that

$$\inf_{i\to\infty} V_{k,\tau_k}(e_i) < V_{k,\tau_k}(e).$$

The lemma follows easily from the preceding lemma, and from it the proof of Theorem 1 is obtained, since

$$V_s(f) = C(n,k,s) \int_{\varphi_n^k} V_{s,\tau_k}(e)\, dm_n^k,$$

where $C(n,k,s)$ is a normalizing factor independent of the set f.

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REFERENCES

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Note: Figure translations are in progress. See original paper for figures.

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