ON ORDERED SPACES

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V. V. FEDORCHUK

ON ORDERED SPACES

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§ 1. An ordered space is a linearly ordered set endowed with the interval topology. An ordered bicompactum B will be called an **ordered bicompact extension** of an ordered space X if: 1) X is dense in B, 2) the set B induces on the set X the original order. The set of ordered bicompact extensions of an ordered space X is a partially ordered set with the order relation induced from the set of all bicompact extensions of the space X.

Theorem 1. The partially ordered set of all ordered bicompact extensions of an ordered space X is a dyadic Boolean algebra.

Corollary. For every ordered space X there exists a maximal ordered bicompact extension bX.

The connection of the maximal ordered bicompact extension bX with the classical constructions of E. Čech and P. S. Aleksandrov (see (1)) is shown by the following assertions:

Theorem 2. Let $f : X \to \prod_\alpha I_\alpha = R^\tau$ be an embedding of the ordered space X into the product of intervals by means of all such continuous functions. Then the closure $[X]_{R^\tau}$ of the set X in R^τ, considered with the order relation induced from R^τ, is the maximal ordered bicompact extension of the space X.

Theorem 3. Let aX be the set of maximal centered regular (in the sense of P. S. Aleksandrov) systems whose elements are intervals of the set X. Then the set aX, endowed with the natural order, is the maximal ordered bicompact extension of the space X.

The maximal ordered bicompact extension bX has characteristic properties analogous to those of the Čech extension: 1) every such continuous function on X extends to bX; 2) any two convex nonintersecting sets closed in X have nonintersecting bX-closures; 3) if an ordered space A is contained as a closed subset in an ordered space X, then $bA = [A]_{bX}$.

Theorem 4. If an ordered space X has weight τ, then bX is the limit of an inverse spectrum of ordered bicompact extensions bX of weight τ.

Let X be an ordered set and $[a, b]$ a jump in the set X. The jump $[a, b]$ will be called two-sided if the points a and b are non-isolated points of the ordered space X.

Definition. An ordered set X without a least and a greatest element will be called **minimal** if X has no two-sided jumps.

Theorem 5. Let X and Y be minimal ordered spaces, and let bX and bY be isomorphic. Then the spaces X and Y are also isomorphic.

Remark. bX and bY may be homeomorphic for minimal spaces X and Y, while the spaces X and Y themselves may fail to be homeomorphic.

Theorem 6. A minimal separable space is metrizable.

Corollary 1. A minimal locally separable paracompact space is metrizable.

Corollary 2. Let X be a minimal separable space. If the remainder $bX \setminus X$ is countable, then bX is a compactum.

Theorem 7. Let X be an ordered space with a point-countable base. Then the following conditions are equivalent: a) the remainder $bX \setminus X$ is countable; b) bX is metrizable.

§ 2. **Definition.** Let X be an ordered set and let $P(X)$ be a proximity space on the set X, inducing on X the interval topology. We shall call the proximity space $P(X)$ an ordered δ-space if the following conditions are satisfied:
1) $x, y \in X,\ x < y \Rightarrow (-\infty, x]\ \bar{\delta}\ [y, +\infty)$;
2) $A, B \subset X,\ A\bar{\delta}B \Rightarrow$ there exists a finite collection of intervals $O_i,\ i = 1, \dots, k$, such that

$$A \subset \bigcup_{i=1}^{k} O_i \subset X \setminus B.$$

Theorem 8. The proximity P, induced on the set X by the proximity of the ordered bicompact extension bX, is ordered.

An analogue of the known theorem of Yu. M. Smirnov (see ([6])) is

Theorem 9. To every ordered δ-space $P(X)$ there corresponds one and only one ordered bicompact extension bX which induces on the set X the proximity P.

Corollary. The partially ordered set of ordered δ-spaces on the set X is a dyadic Boolean algebra.

Theorem 10. Let

$$f : P(X) \to \prod I_\alpha = R^\tau$$

be an embedding of the ordered δ-space $P(X)$ into a product of intervals by means of all similar δ-continuous functions. Then the closure $[X]_{R^\tau}$ of the set X in R^τ will be an ordered bicompact extension of the space X, generating the ordered δ-space $P(X)$.

Theorem 11. The completion of an ordered δ-space is an ordered bicompactum ($cP = uP$).

From Theorem 11 and known results of Yu. M. Smirnov (see ([7])) there follow the corollaries.

Corollary 1. An ordered δ-space has a unique uniform structure compatible with it.

Corollary 2. If $P(X)$ is a complete ordered δ-space, then for any ordered δ-space $P(Y) \subset P(X)$ we have

$$cP(Y) = [P(Y)]_{P(X)}.$$

Corollary 3. If an ordered δ-space is metrizable, then it has a countable base.

Theorem 12. In order that a proximity P on an ordered set X be ordered, it is necessary and sufficient that the proximity P be fully bounded and satisfy property 1) of ordered proximity.

§ 3. In accordance with ([5]), we shall call a multivalued mapping $f : X \to Y$ of an ordered space X onto an ordered space Y multivalued, similar, and irreducible if there exists an ordered space Z and such similar irreducible single-valued mappings $f_X : Z \to X$ and $f_Y : Z \to Y$ that $f = f_Y f_X^{-1}$. An analogue of the theorem of V. I. Ponomarev ([5]) is:

Theorem 13. Let X be an ordered space. There exists an ordered space aX (the ordered absolute), which is irreducibly and similarly mapped onto the ordered space X by means of the natural projection π_X. If an ordered space Z is irreducibly and similarly mapped onto the space aX by a mapping g, then g is an isomorphism. If the space X, by means of a many-valued irreducible similar mapping f, is mapped onto an ordered space Y, then there exists an isomorphism $h : aX \to aY$ of ordered absolutes such that $f = \pi_Y h \pi_X^{-1}$.

Definition. An ordered space X will be called **order extremally disconnected** if, for every open interval $V \subset X$, its closure $[V]$ is open.

Theorem 14. In order that an ordered space X be an ordered absolute, it is necessary and sufficient that X be an order extremally disconnected space.

The following theorem is an analogue of the theorem of C. Iliadis ([2]).

Theorem 15. The ordered bicompact extension bX of the ordered absolute X is an ordered absolute if and only if $bX = \beta X$.

Corollary. $\beta aX = abX$.

Theorem 16. If X and Y are minimal ordered spaces and their ordered absolutes aX and aY are isomorphic, then X and Y are also isomorphic.

§ 4. Let N be a well-ordered set, and let $X_\alpha (\alpha \in N)$ be linearly ordered sets. As is known (see, for example, ([4])), the ordered product

$$\mathbf{P}_{\alpha \in N} X_\alpha$$

is the set of all such mappings f, defined on N, that $f(\alpha) \in X_\alpha$ for every $\alpha \in N$, ordered in the following way: $f < g \iff$ there exists an index $\alpha_0 \in N$ such that $f(\alpha) = g(\alpha)$ for all $\alpha < \alpha_0$ and $f(\alpha_0) < g(\alpha_0)$. It is easily verified that the set

$$\mathbf{P}_{\alpha \in N} X_\alpha$$

is linearly ordered. If all the sets X_α coincide with a single set X, then the corresponding ordered product is called an ordered power and is denoted by ${}^N X$. If τ is a cardinal number, then by ${}^\tau X$ one denotes the ordered power ${}^N X$, where the well-ordered set N has order type $\omega(\tau)$ of the least ordinal number of cardinality τ.

Novák proved ([4]) that an ordered power of an ordered continuum is an ordered continuum. His result can be generalized.

Theorem 17. Let N be a well-ordered set, and let, for every $\alpha \in N$, B_α be an ordered bicompactum. Then their ordered product

$$B = \mathbf{P}_{\alpha \in N} B_\alpha$$

is also an ordered bicompactum; moreover, B is a continuum if and only if all B_α are continua. If N has no greatest element, then $\dim B = 0$ if and only if there exists in N a cofinal subset N' such that the bicompactum B_α is disconnected for every $\alpha \in N'$.

The following theorem shows that the class of ordered bicompacta is obtained from an ordered pair of points by means of the operations of ordered product and mappings under which the inverse images of points are intervals of an ordered set.

Theorem 18. Let B be an ordered bicompactum of weight τ. Then B is a similar continuous image of the ordered bicompactum ${}^\tau D$.

C. Marlepsch and P. Papic, in paper ([3]), called a mapping $f : K \to X$ of an ordered bicompactum K light in the order sense if, for every point $x \in X$, every order component of the inverse image $f^{-1}(x)$ consists of only one point. In that paper they proved that the degree of cellularity $c(K)$ (the upper bound of the cardinalities of disjoint systems of nonempty open–

closed sets) cannot be lowered under a quasi-open and order-light mapping $f : K \to X$ of an ordered bicompactum K, if the bicompactum K consists of an infinite number of points, and he posed the problem of proving this theorem without using the quasi-openness of the mapping f. Below a negative solution of this problem is given.

Theorem 19. Let τ be such a cardinal number that $\mathfrak{m} < \tau \Rightarrow 2^{\mathfrak{m}} \leq \tau$. There exist an ordered bicompactum B_τ such that $c(B_\tau) = \tau$, an ordered bicompactum K_τ such that $c(K_\tau) = 2^\tau$, and an order-light mapping $f_\tau : K_\tau \to B_\tau$.

$B_\tau = a^\tau D$ is the ordered absolute of the ordered bicompactum ${}^\tau D$. K_τ is obtained from B_τ by filling each jump $[a, b]$ with a pair of points a', b' with the order relation $a < a' < b' < b$.

The mapping f_τ is defined by the equalities

$$f_\tau(\{a, b'\}) = a, \qquad f_\tau(\{a', b\}) = b.$$

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Moscow State University
named after M. V. Lomonosov

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Note: Figure translations are in progress. See original paper for figures.