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**Abstract**

**Full Text**

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*MATHEMATICS*

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## ON THE BIRTH OF PERIODIC MOTION FROM A TRAJECTORY ISSUING FROM AN EQUILIBRIUM STATE OF SADDLE-SADDLE TYPE AND RETURNING TO IT

*(Presented by Academician A. Yu. Ishlinskii, November 22, 1965)*

1. Consider a system of differential equations of order  $(m + n + 1)$

$$\begin{aligned} dx/dt &= P(x, y, z, \mu), & dy/dt &= Q(x, y, z, \mu), \\ dz/dt &= R(x, y, z, \mu), \end{aligned} \tag{1}$$

where  $P(P_1, P_2, \dots, P_m)$ ,  $Q(Q_1, Q_2, \dots, Q_n)$ , and  $R$  are analytic or sufficiently smooth functions of the variables  $x(x_1, x_2, \dots, x_m)$ ,  $y(y_1, y_2, \dots, y_n)$ ,  $z$ , and of the parameter  $\mu$  in some domain  $G \times (-\mu_0, \mu_0)$ . Suppose that for  $\mu = 0$  system (1) has an equilibrium state  $O \in G$ , at which the characteristic equation has  $m$  roots with negative real parts,  $n$  roots with positive real parts, and one zero root.

By means of a linear nonsingular change of variables, system (1) can be brought to the form

$$\begin{aligned} dx/dt &= Ax + P_0(x, y, z, \mu), & dy/dt &= By + Q_0(x, y, z, \mu), \\ dz/dt &= R_0(x, y, z, \mu), \end{aligned} \tag{2}$$

where  $P_0, Q_0, R_0$ , for  $\mu = 0$ , vanish at  $O(0, 0, 0)$  together with their first derivatives;  $A$  is a matrix of order  $m$ , whose characteristic polynomial has roots  $\lambda_1, \lambda_2, \dots, \lambda_m$  with negative real parts, and  $B$  is a matrix of order  $n$ , whose characteristic polynomial has roots  $\gamma_1, \gamma_2, \dots, \gamma_n$  with positive real parts. Let

$$x = u(z), \quad y = v(z)$$

be a solution of the system

$$Ax + P_0(x, y, z, 0) = 0, \quad By + Q_0(x, y, z, 0) = 0$$

for sufficiently small  $z$ . Suppose that in the expansion

$$R_0(u(z), v(z), z, 0) = l_k z^k + l_{k+1} z^{k+1} + \dots \quad (3)$$

the first nonzero quantity  $l_k$  has an even index and, for definiteness, is positive.

The behavior of phase trajectories in a neighborhood of such a complex equilibrium state was studied in the work of R. M. Mintz <sup>(1)</sup>; she called the equilibrium state itself an equilibrium state of type  $C$ . This equilibrium state is formed by the coalescence of  $2k$  simple equilibrium states of saddle type; therefore, it seems to us more natural to call  $O$  a complex equilibrium state of saddle-saddle type.

Let us describe the behavior of integral curves in a neighborhood of  $O$ . It is characterized first of all by the existence of an  $(n + m)$ -dimensional separatrix surface  $\pi$ , passing through  $O$  and tangent to the  $x$ - and  $y$ -axes. This surface divides a neighborhood of  $O$  into two regions. In the region containing the half-axis  $z \leq 0$ , there exists a unique  $(m + 1)$ -dimensional integral  $O^+$ -semisurface (a semisurface consisting of  $O^+$ -curves), which we shall denote by  $\pi^+$ . In the region containing the half-axis  $z \geq 0$ , there exists a unique  $(n + 1)$ -dimensional integral  $O^-$ -semisurface (a semisurface consisting of  $O^-$ -curves), which we shall denote by  $\pi^-$ . The boundaries of the semisurfaces  $\pi^+$  and  $\pi^-$ , which we shall denote by  $\Pi^+$  and  $\Pi^-$ , are manifolds of dimensions  $m$  and  $n$ , respectively. They lie in  $\pi$  and intersect at the point  $O$ . All integral curves, except those that lie in  $\pi^+$  and  $\pi^-$ , pass by  $O$ .

The purpose of the present paper is to prove the following theorem.

**Theorem.** *Suppose that for  $\mu = 0$  system (1) has the following properties: 1) there exists a trajectory  $\Gamma_0$  issuing from  $O$  and returning to it again as  $t \rightarrow +\infty$ ; 2)  $\Gamma_0$  does not lie in  $\Pi^+$  and  $\Pi^-$ ; 3)  $\pi^+$  and  $\pi^-$  intersect along  $\Gamma_0$ , without being tangent.\**

*Then, for all sufficiently small  $\mu \neq 0$  for which the composite equilibrium state of saddle-saddle type disappears, a unique periodic motion of saddle type is born from  $\Gamma_0$ .*

The conditions for the disappearance of an equilibrium state of saddle-saddle type are completely analogous to the conditions for the disappearance of a composite equilibrium state of saddle-node type <sup>(4)</sup>.

The proof of the theorem is based on the method of point mappings and consists in proving the existence of a fixed point of a mapping  $T$  of the plane  $z = -d$  ( $d > 0$ ) into itself.

We shall seek the mapping  $T$  in the form  $T = T_1 T_0$ , where  $T_0$  is the mapping of the plane  $z = -d$  into  $z = d$  in a neighborhood of the origin, and  $T_1$  is the mapping of  $z = +d$  into  $z = -d$  in a neighborhood of the global arc  $\Gamma_0$ .

**2. Construction of the mapping  $T_0$ .** Let us first consider system (2) for  $\mu = 0$ . As follows from the results of V. A. Pliss <sup>(2,3)</sup>, in a neighborhood of the origin  $O$  there exist two integral surfaces

$$y = \psi(x, z), \quad x = \varphi(y, z), \quad (4)$$

whose dimensions are  $m + 1$  and  $n + 1$ , respectively. The functions  $\psi$  and  $\varphi$  are sufficiently smooth and vanish at the origin together with their first derivatives. On these surfaces  $O$  will be a composite equilibrium state of saddle-node type. From the uniqueness of the integral  $O^+$ -semisurface  $\pi^+$  and the  $O^-$ -semisurface  $\pi^-$  it follows that, in the region to the left of  $\pi$ , the surface  $y = \psi(x, z)$  coincides with  $\pi^+$ , while in the region to the left of  $\pi$  the surface  $x = \varphi(y, z)$  coincides with  $\pi^-$ .

By means of a certain sufficiently smooth change of variables in a sufficiently small neighborhood of the origin, system (2) can be written in the form

$$\begin{aligned} d\xi/dt &= A\xi + P'(\xi, \eta, z)\xi + \mu\alpha(z, \mu)R_0(z, \mu), \\ d\eta/dt &= B\eta + Q'(\xi, \eta, z)\eta + \mu\beta(z, \mu)R_0(z, \mu), \\ dz/dt &= R_0(z, \mu) + R'_1(\xi, \eta, z, \mu)\xi + R'_2(\xi, \eta, z, \mu)\eta, \end{aligned} \quad (5)$$

where  $P'$ ,  $Q'$ ,  $R'_1$ ,  $R'_2$  are small for sufficiently small  $\xi, \eta, z, \mu$ , while  $R_0(z, 0) = l_k z^k + \dots$ . In the new variables the equations of the surfaces (4) will be, respectively,  $\eta = 0$  and  $\xi = 0$ , and the equations of  $\Gamma_0$  will be  $\xi = 0$ ,  $\eta = 0$ . It is obvious that the condition for the disappearance of the saddle-saddle will consist in the fulfillment of the inequality  $R_0(z, \mu) > 0$ . Suppose, for definiteness, that  $(0, \bar{\mu}_0)$  is an interval of values of  $\mu$  for which this inequality is satisfied. A value of  $\mu$  from this interval will be denoted by  $\bar{\mu}$ .

\* From the works of V. A. Pliss <sup>(2,3)</sup> it follows that  $\pi^+$  and  $\pi^-$  are smooth surfaces.

Introduce into consideration the closed surface  $S_{\bar{\mu}}$ , consisting of the surfaces

$$\begin{aligned} V \equiv \|\xi\|^2 - R_0^2(z, \bar{\mu}) &= 0, & W \equiv \|\eta\|^2 - R_0^2(z, \bar{\mu}) &= 0, \\ S_0 : z = -d, & & S_1 : z = d. & \end{aligned} \quad (6)$$

It is easy to see that inside  $S_{\bar{\mu}}$ , for sufficiently small  $d$  and sufficiently small  $\mu$ , the condition  $dz/dt > 0$  is satisfied. After eliminating the time  $t$ , system (5) inside  $S_{\bar{\mu}}$  can be written in the form

$$\frac{d\xi}{dz} = \frac{A\xi + P_1(\xi, \eta, z, \bar{\mu})\xi + \mu\alpha(z, \bar{\mu})R_0(z, \bar{\mu})}{R_0(z, \bar{\mu})},$$

$$\frac{d\eta}{dz} = \frac{B\eta + Q_1(\xi, \eta, z, \bar{\mu})\eta + \mu\beta(z, \bar{\mu})R_0(z, \bar{\mu})}{R_0(z, \bar{\mu})},$$

where  $P_1$  and  $Q_1$  are small inside  $S_{\bar{\mu}}$  for sufficiently small  $d$  and  $\bar{\mu}_0$ .

**Lemma 1.** For every pair  $\xi_0$  and  $\eta_1$  satisfying the conditions

$$\|\xi_0\| < a < R_0(-d, 0), \quad \|\eta_1\| < b < R_0(d, 0),$$

there exists a solution of system (7)

$$\xi(z) = \xi(z, \xi_0, \eta_1, \bar{\mu}), \quad \eta(z) = \eta(z, \xi_0, \eta_1, \bar{\mu}), \quad (8)$$

lying inside  $S_{\bar{\mu}}$  and satisfying the conditions

$$\xi(-d) = \xi_0, \quad \eta(d) = \eta_1,$$

and moreover

$$\|\xi(z)\| < \|\xi_0\| \exp \left[ \lambda \int_{-d}^z \frac{dz}{R_0(z, \bar{\mu})} \right] + \bar{K}\mu,$$

$$\|\eta(z)\| < \|\eta_1\| \exp \left[ \gamma \int_{+d}^z \frac{dz}{R_0(z, \bar{\mu})} \right] + \bar{K}\mu, \quad (9)$$

where

$$\max_{1 \leq i \leq m} \operatorname{Re} \lambda_i < \lambda < 0, \quad 0 < \gamma < \min_{1 \leq i \leq n} \operatorname{Re} \gamma_i,$$

and  $\bar{K}$  is some fixed constant.

**Lemma 2.** The estimates

$$\left\| \frac{\partial \xi}{\partial \xi_0} \right\| + \left\| \frac{\partial \eta}{\partial \xi_0} \right\| < \exp \left[ \lambda \int_{-d}^z \frac{dz}{R_0(z, \bar{\mu})} \right],$$

$$\left\| \frac{\partial \xi}{\partial \eta_1} \right\| + \left\| \frac{\partial \eta}{\partial \eta_1} \right\| < \exp \left[ \gamma \int_{+d}^z \frac{dz}{R_0(z, \bar{\mu})} \right]$$

hold for all  $z \in [-d, d]$  for sufficiently small  $d$  and  $\bar{\mu}_0$ .

Consider the integral curve (8). Obviously, specifying  $(\xi_0, \eta_1)$  makes it possible to determine uniquely

$$\xi_1 = \xi(d, \xi_0, \eta_1, \bar{\mu}) \quad \text{and} \quad \eta_0 = \eta(-d, \xi_0, \eta_1, \bar{\mu}).$$

The mapping of the points  $M_0(\xi_0, \eta_0) \in S_0$  into the points  $M_1(\xi_1, \eta_1) \in S_1$  obtained in this way will be denoted by  $T_0$ . At the same time, note that, since

$$\lim_{\bar{\mu} \rightarrow 0} \int_{-d}^d \frac{dz}{R_0(z, \bar{\mu})} = +\infty,$$

we have

$$\|\xi_1\| + \|\eta_0\| + \|\partial\xi_1/\partial\xi_0\| + \|\partial\eta_0/\partial\xi_0\| + \|\partial\xi_1/\partial\eta_1\| + \|\partial\eta_0/\partial\eta_1\| \rightarrow 0 \quad (10)$$

as  $\bar{\mu} \rightarrow 0$ .

**3. Construction of the mapping  $T_1$ .** In the variables  $\xi, \eta, z$ ,  $\Gamma_0$  intersects  $S_0$  and  $S_1$  at the points  $M_0^0(0, 0)$  and  $M_1^1(0, 0)$ . As is known, for sufficiently small

$$0 < a < \frac{1}{2}R_0(-d, 0)$$

one can indicate such a

$$0 < b < \frac{1}{2}R_0(d, 0)$$

that, for all sufficiently small  $\mu$ , the trajectories passing through the points  $M_1 \in S_1$ , lying in the  $b$ -neighborhood of  $M_1^1$ , will intersect  $S_0$  at points  $\bar{M}_0(\bar{\xi}_0, \bar{\eta}_0)$ , lying in the  $a$ -neighborhood of  $M_0^0$ . The resulting map-

We shall denote the mapping by  $T_1$ . In the variables under consideration it will be written in the form

$$\begin{aligned} \bar{\xi}_0 &= f_1(\xi_1, \eta_1, \mu) = A_1(\mu) + A_{11}(\mu)\xi_1 + A_{12}(\mu)\eta_1 + \dots, \\ \bar{\eta}_0 &= g_1(\xi_1, \eta_1, \mu) = A_2(\mu) + A_{21}(\mu)\xi_1 + A_{22}(\mu)\eta_1 + \dots \end{aligned} \quad (11)$$

Since, by assumption,  $\pi^+$  and  $\pi^-$  intersect along  $\Gamma_0$  without tangency,  $|A_{22}(0)| \neq 0$ .

4. The completion of the proof of the theorem is reduced to proving the existence for the mapping  $T = T_1 T_0$  of a fixed point  $M^*(\xi_0^*, \eta_0^*)$ , tending to  $M_0^0$  as  $\mu \rightarrow 0$ . Denote by  $\sigma(a, b)$  the image of the domain  $\|\xi_0\| < a, \|\eta_1\| < b$  under the mapping  $(\xi_0, \eta_1) \rightarrow (\xi_0, \eta_0)$ , where  $\eta_0 = \eta(-d, \xi_0, \eta_1, \mu)$ . It is easy to see, by virtue of the choice of  $a$  and  $b$ , that the mapping  $T$  is defined on  $\sigma(a, b) \in S_0$  and maps  $\sigma(a, b)$  into  $S_0$ .

Let us consider the equations for finding the coordinates of a fixed point. In the variables  $\xi_0^*, \eta_1^*$  they will be written in the form

$$\begin{aligned} F(\xi_0^*, \eta_1^*, \bar{\mu}) &\equiv \xi_0^* - f_1(\xi(d, \xi_0^*, \eta_1^*, \bar{\mu}), \eta_1^*, \bar{\mu}) = 0, \\ G(\xi_0^*, \eta_1^*, \bar{\mu}) &= \eta(-d_0, \xi_0^*, \eta_1^*, \bar{\mu}) - g_1(\xi(d, \xi_0^*, \eta_1^*, \bar{\mu}), \eta_1^*, \bar{\mu}) = 0. \end{aligned} \quad (12)$$

Since, using (10),  $F$  and  $G$  can be extended continuously to the boundary  $\bar{\mu} = 0$ , together with their derivatives, and since for  $\bar{\mu} = 0$  the system under consideration has the unique solution  $\xi_0^* = 0, \eta_1^* = 0$ , with Jacobian, at  $\xi_0^* = \eta_1^* = \bar{\mu} = 0$ , equal to  $-|A_{22}(0)|$ , it follows that for sufficiently small  $\bar{\mu} > 0$  we obtain that the system (12) has a unique solution  $\xi_0^*(\bar{\mu}), \eta_1^*(\bar{\mu})$ , where  $\xi_0^*(\bar{\mu})$  and  $\eta_1^*(\bar{\mu})$  tend to zero together with  $\bar{\mu}$ . Consequently, the mapping  $T$  has a fixed point  $M^*(\xi_0^*, \eta_1^*)$ , tending to  $M_0^0$  as  $\bar{\mu} \rightarrow 0$ . We shall prove that  $M^*$  is a fixed point of saddle type. Obviously, the characteristic equation, using the estimates (10), can be written in the form

$$\alpha_n(\bar{\mu})z^{m+n} + \dots + \alpha_1(\bar{\mu})z^{m+1} + (|A_{22}(0)| + O(\bar{\mu}))z^m + \beta_m(\bar{\mu})z^{m-1} + \dots + \beta_1(\bar{\mu}) = 0, \quad (13)$$

where  $\alpha_i(\bar{\mu})$  and  $\beta_i(\bar{\mu})$  tend to zero as  $\bar{\mu} \rightarrow 0$ . Since the coefficient of  $z^m$  is not equal to zero for sufficiently small  $\bar{\mu}$ , we obtain that equation (13) has  $m$  roots with moduli less than unity and  $n$  roots with moduli greater than unity.

From the known relation between fixed points of a mapping and periodic solutions we obtain that from  $\Gamma_0$  there is born a unique periodic motion of saddle type.

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*Note: Figure translations are in progress. See original paper for figures.*

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