THE DIRICHLET PROBLEM FOR STRONGLY COUPLED SYSTEMS OF ELLIPTIC TYPE

MATHEMATICS

1966

SovietRxiv

Abstract

Full Text

UDC 517.946

MATHEMATICS

NGUEN TKHYA KHOSH

THE DIRICHLET PROBLEM FOR STRONGLY COUPLED SYSTEMS OF ELLIPTIC TYPE

(Presented by Academician M. A. Lavrent' ev, 28 I 1966)

Let an elliptic system be given

$$A\frac{\partial^2 u}{\partial x^2} + 2B\frac{\partial^2 u}{\partial x\,\partial y} + C\frac{\partial^2 u}{\partial y^2} = 0, \tag{1}$$

$$\det[A + 2B\lambda + C\lambda^2] \neq 0$$

for real λ; $\det C \neq 0$; A, B, C are given constant matrices; u is the unknown vector with components $u_1, u_2, \dots, u_n$.

For a system of the form (1), A. V. Bitsadze ([2]) introduced the concepts of weak and strong coupling, which proved useful in the study of linear boundary-value problems for the indicated system. As is known, the Dirichlet problem for weakly coupled systems is Fredholm ([6]) (see also ([4])). In ([2]) examples are given of strongly coupled systems of two equations for which the Dirichlet problem ceases to be Fredholm and Noetherian. In ([7]) the Dirichlet problem for a strongly coupled system in the disk $|z| < 1$ was investigated in the case when i is an n-fold root of the characteristic equation

$$\det[A + 2B\lambda + C\lambda^2] = 0. \tag{2}$$

The present paper is devoted to the study of the Dirichlet problem for strongly coupled systems (1) of two equations in the general case.

Introduce the differential operators

$$L_1(w) = \partial^2 w/\partial \bar{z}^2 + \sigma\,\partial^2 w/\partial z\,\partial \bar{z} + \pi\,\partial^2 w/\partial z^2,$$

$$\bar{L}_1(w) = \partial^2 w/\partial z^2 + \bar{\sigma}\,\partial^2 w/\partial z\,\partial \bar{z} + \bar{\pi}\,\partial^2 w/\partial \bar{z}^2,$$

where $z = x + iy$, $\bar{z} = x - iy$, $w = u_1 + iu_2$.

Theorem 1. In order that system (1), written in complex form

$$L(w) = a\,\partial^2 w/\partial z^2 + 2b\,\partial^2 w/\partial z\,\partial\bar{z} + c\,\partial^2 w/\partial\bar{z}^2 +$$

$$+d\,\partial^2\bar{w}/\partial z^2 + 2e\,\partial^2\bar{w}/\partial z\,\partial\bar{z} + f\,\partial^2\bar{w}/\partial\bar{z}^2 = 0, \quad (3)$$

be elliptic and strongly coupled, it is necessary and sufficient that it have the form

$$L(w) = \alpha L_1(w) + \beta L_1(\bar{w}) + \gamma \bar{L}_1(\bar{w}) + \delta \bar{L}_1(w) = 0, \quad (4)$$

where $\alpha, \beta, \gamma, \delta, \sigma, \pi$ are complex numbers satisfying the conditions:

$$\begin{aligned} &1. \quad \alpha\delta = \beta\gamma, \\ &2. \quad |\alpha|^2 - |\beta|^2 + |\gamma|^2 - |\delta|^2 \neq 0; \\ &3. \quad (|\sigma|^2 - |\sigma^2 - 4\pi|)/2 < 1 + |\pi|^2. \end{aligned} \quad (5)$$

Theorem 2. In order that system (3) be elliptic and strongly coupled, it is necessary and sufficient that the conditions hold:

1. $(f\bar{f} - d\bar{d})(\bar{b}c - b\bar{a}) - (c\bar{c} - a\bar{a})(\bar{e}f - e\bar{d}) = 0.$

2. $$\big[(c\bar{c} + a\bar{a}) - (d\bar{d} + f\bar{f})\big]\big[(c\bar{c} - a\bar{a})^2 + (d\bar{d} - f\bar{f})^2\big] \neq 0.$$

3. $$|r + p\bar{r}|^2 + \big|(r + p\bar{r})^2 - 4p\big| < 2(1 + |p|^2).$$

Here p is a root of the quadratic equation

$$p^2(\bar{a}c - d\bar{f}) + p\{(a\bar{a} + c\bar{c}) - (d\bar{d} + f\bar{f})\} + \bar{c}a - \bar{f}d = 0,$$

$$r = \begin{cases} 2\,\dfrac{\bar{b}c - b\bar{a}}{c\bar{c} - a\bar{a}}, & \text{if } c\bar{c} \neq a\bar{a}, \\[2ex] 2\,\dfrac{e\bar{f} - \bar{e}d}{f\bar{f} - d\bar{d}}, & \text{if } f\bar{f} \neq d\bar{d}. \end{cases}$$

When condition (5) is satisfied, the strongly coupled elliptic system

$$L_1(W) = 0 \tag{6}$$

will below be called **fundamental**.

In the notation $\sigma = \nu_1 + \nu_2$, $\pi = \nu_1 \nu_2$, condition (5) is equivalent to the inequalities $|\nu_1| < 1$, $|\nu_2| < 1$ or $|\nu_1| > 1$, $|\nu_2| > 1$. The roots λ_1 and λ_2 ($\operatorname{Im}\lambda_1 > 0$ and $\operatorname{Im}\lambda_2 > 0$) of the characteristic equation (2) are related to ν_1 and ν_2 by the formulas

$$\lambda_1 = -i\,\frac{\nu_1 + 1}{\nu_1 - 1}, \qquad \lambda_2 = -i\,\frac{\nu_2 + 1}{\nu_2 - 1}.$$

The systems

$$M(w^*) = \bar{\alpha} L_1(w^*) + \beta L_1(\bar{w}^*) + \gamma \bar{L}_1(\bar{w}^*) + \bar{\delta} L_1(w^*) = 0, \tag{7}$$

$$\bar{L}_1(W^*) = 0 \tag{8}$$

are the systems Lagrange-adjoint (in the variables (x, y)) respectively to the systems (4) and (6).

The general solutions of the fundamental system (6) and its adjoint (8), for $\lambda_1 = \lambda_2$, have respectively the form

$$W = \overline{(z - \nu\bar{z})}\,\Phi(z - \nu\bar{z}) + \Psi(z - \nu\bar{z}), \tag{9}$$

$$W^* = (z - \nu\bar{z})\,\overline{\Phi^*(z - \nu\bar{z})} + \overline{\Psi^*(z - \nu\bar{z})}, \tag{10}$$

and for $\lambda_1 \neq \lambda_2$,

$$W = \Phi(z - \nu_1\bar{z}) + \Psi(z - \nu_2\bar{z}), \tag{11}$$

$$W^* = \overline{\Phi^*(z - \nu_1\bar{z})} + \overline{\Psi^*(z - \nu_2\bar{z})}; \tag{12}$$

here $\Phi, \Psi, \Phi^*, \Psi^*$ are arbitrary holomorphic functions of their arguments.

Theorem 3. There exist complex numbers $\varepsilon_1, \varepsilon_2, \tau_1, \tau_2$, satisfying the conditions $|\varepsilon_1| \neq |\varepsilon_2|$, $|\tau_1| \neq |\tau_2|$, such that

$$w = \varepsilon_1 W + \varepsilon_2 \bar{W}, \tag{13}$$

$$w^* = \tau_1 W^* + \tau_2 \bar{W}^*, \tag{14}$$

where w, w^*, W, W^* are the general solutions of the systems (4), (7), (6), (8), respectively.

Let $\mathcal{D}$ be a simply connected domain bounded by a rectifiable curve Γ; $\mathcal{D}_\zeta, \mathcal{D}_{\zeta_1}, \mathcal{D}_{\zeta_2}$ are the images of the domain $\mathcal{D}$ under the transformations $\zeta = z - \nu\bar{z}$; $\zeta_1 = z - \nu_1\bar{z}$, $\zeta_2 = z - \nu_2\bar{z}$. Solutions W and W^* of the systems (6) and (8) will be called **regular** if in the expressions (9), (10), (11), (12) the functions $\Phi(\zeta), \Psi(\zeta), \Phi^*(\zeta), \Psi^*(\zeta) \in E_2(\mathcal{D}_\zeta)$; $\Phi(\zeta_1)$, $\partial\Phi^*/\partial\zeta_1 \in E_2(\mathcal{D}_{\zeta_1})$, $\Psi(\zeta_2)$, $\partial\Psi^*/\partial\zeta_2 \in E_2(\mathcal{D}_{\zeta_2})$ (for the definition of the class E_2, see [8]). In accordance with this, solutions w and w^* of the systems (4) and (7) will be called regular–

are, if the functions W and W^* in (13) and (14) are regular solutions of systems (6) and (8).

Definition. By the Dirichlet problem $D(\nu_1, \nu_2)$ in the case $\nu_1 \neq \nu_2$ (respectively, $D(\nu)$ in the case $\nu_1 = \nu_2 = \nu$) for system (4) we shall mean the problem of finding regular solutions of this system satisfying the condition

$$w\big|_\Gamma = f(s), \tag{15}$$

where $f(s) \in L_2(\Gamma)$. The homogeneous problem corresponding to (4), (15) will be denoted by $D_0(\nu_1, \nu_2)$ $(D_0(\nu))$.

By the homogeneous problem $D_0^*(\nu_1, \nu_2)$, $\nu_1 \neq \nu_2$ $(D_0^*(\nu), \nu_1 = \nu_2 = \nu)$, adjoint to $D(\nu_1, \nu_2)$ (to $D(\nu)$), we shall mean the problem of finding regular solutions of system (7) satisfying the condition

$$w^*\big|_\Gamma = 0. \tag{16}$$

From Green' s formula one obtains the necessary solvability condition for the problem $D(\nu_1, \nu_2)$ $(D(\nu))$ in the form

$$\int_\Gamma \left[f Q(\overline{w^*}) + \overline{f}\, \overline{Q(w^*)} \right] ds = 0, \tag{17}$$

where w^* are regular solutions of the problem $D_0^*(\nu_1, \nu_2)$, and

$$Q(\overline{w^*}) = \overline{\mu_2\, Q_1(\overline{\theta_2\omega^* + \theta_1\omega^*})} + \mu_1 Q_1(\overline{\theta_2\omega^* + \theta_1\omega^*}),$$

with $|\mu_1| \neq |\mu_2|$, $|\theta_1| \neq |\theta_2|$ being definite constants depending only on the coefficients of the equations, and

$$Q_1(\ ) = \nu_1\nu_2 \frac{d\bar{z}}{ds}\frac{\partial}{\partial\bar{z}} - (\nu_1+\nu_2)\frac{dz}{ds}\frac{\partial}{\partial z} - \frac{dz}{ds}\frac{\partial}{\partial z}.$$

If condition (17) also proves sufficient for the solvability of the problem $D(\nu_1,\nu_2)$ ($D(\nu)$), then we shall call this problem **normally solvable in the sense of Hausdorff** (this term was introduced by A. V. Bitsadze ([3])).

Theorem 4. A. *The spaces of solutions of the homogeneous problems $D_0(\nu_1,\nu_2)$ ($D_0(\nu)$) of systems (4) and (6) are simultaneously either zero-dimensional or infinite-dimensional.*

B. *From the normal solvability of the problem $D(\nu_1,\nu_2)$ ($D(\nu)$) for system (4) there follows the normal solvability of this problem for the fundamental system (6), and conversely.*

This theorem answers the question posed in ([5]): the homogeneous Dirichlet problem for the equation

$$\frac{\partial}{\partial z}\left(\frac{\partial w}{\partial z} + \lambda\frac{\partial w}{\partial\bar{z}}\right) = 0, \qquad 0 < |\lambda| < 1,$$

cannot have a finite number of nonzero linearly independent solutions.

By virtue of Theorem 4, in studying the problem $D(\nu_1,\nu_2)$ ($D(\nu)$) for the strongly coupled system (3), it suffices to restrict ourselves to the study of this problem for the fundamental system (6).

The Dirichlet problem $D(\nu)$ for the fundamental system (6) is the Dirichlet problem for the system

$$w_{z_\lambda\bar{z}_\lambda} = 0, \qquad z_\lambda = x + \lambda y,$$

and it was studied in ([9]). Thus, below we shall consider only the problem $D(\nu_1,\nu_2)$, $\nu_1 \neq \nu_2$.

Theorem 5. The homogeneous problem $D_0(\nu_1,\nu_2)$ in the disk has an infinite set of linearly independent solutions if and only if ν_1/ν_2 ($\nu_2 \neq 0$) is equal to a root of unity.

In studying the problem $D(\nu_1,\nu_2)$ in an ellipse, without loss of generality one may consider only the case when the equation of the ellipse has the form

$$mz^2 + \bar{m}\bar{z}^2 + 2z\bar{z} = n^2, \qquad |m| < 1, \tag{18}$$

where n is a real number.

Let $p = 1 + \sqrt{1-|m|^2}$.

Theorem 6. In a finite domain bounded by the ellipse (18), the homogeneous problem $D_0(\nu_1, \nu_2)$ has an infinite set of linearly independent solutions if and only if

$$\frac{\bar{m} + \nu_1 p}{p + \nu_1 m} \frac{p + \nu_2 m}{m + \nu_2 p} \quad (\neq 1)$$

is a root of unity.

Theorem 7. The problem $D(\nu_1, \nu_2)$ is not normally solvable (in the sense of Hausdorff) in a domain with analytic boundary.

Theorem 8. In a finite domain $\widetilde{D}$ bounded by a Lyapunov curve with exponent $> 1/2$, the problem $D(\nu_1, \nu_2)$ is not Noetherian.

The proof of the theorems formulated above is carried out by reducing the problem $D(\nu_1, \nu_2)$ to an equivalent Fredholm integral equation of the first kind.

Novosibirsk State
University

Received
26 I 1966

REFERENCES

[1] A. V. Bitsadze, *Uspekhi Mat. Nauk*, 3, 6 (28) (1948).
[2] A. V. Bitsadze, *Equations of Mixed Type*, 1959.
[3] A. V. Bitsadze, *Dokl. Akad. Nauk*, 164, No. 6 (1965).
[4] A. V. Bitsadze, *Boundary-Value Problems for Elliptic Equations of the Second Order*, 1966.
[5] N. E. Tovmasyan, Materials for the joint Soviet-American symposium on partial differential equations. August 1963, Novosibirsk.
[6] E. V. Zolotareva, *Dokl. Akad. Nauk*, 145, No. 4 (1962).
[7] E. V. Zolotareva, Candidate’ s dissertation, Novosibirsk, 1963.
[8] I. I. Privalov, *Boundary Properties of Analytic Functions*, 1950.
[9] Nguyen Tkha Hop, *Differential Equations*, No. 2 (1966).

Note: Figure translations are in progress. See original paper for figures.